Singular behaviour of rotationally symmetric surfaces of codimension two evolving under mean curvature flow

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Zusammenfassung

In dieser Arbeit wird der mittlere Krümmungsfluss für in den $\mathbb{R}^4$ immergierte, rotationssymmetrische Flächen betrachtet. Dazu sei $c_0 : S^1 \to \mathbb{R}^3_{>0}$ eine glatte, geschlossene Kurve im Halbraum $\mathbb{R}^3_{>0} := \left\{ y \in \mathbb{R}^3 ; y^1 > 0 \right\}$. Rotation der Kurve in $\mathbb{R}^4$ liefert eine Immersion $F_0 : S^1 \times S^1 \to \mathbb{R}^4$, so dass das Bild M_0 invariant unter allen Rotationen der $y^1 y^4$-Ebene ist:

$$M_0 := \left\{ \left(c^1 \left(x^1 \right) \cos x^2, c^2 \left(x^1 \right), c^3 \left(x^1 \right), c^1 \left(x^1 \right) \sin x^2 \right); x^1, x^2 \in S^1 \right\}.$$

Es sei F_t die Ein-Parameter-Familie von Lösungen des mittleren Krümmungsflusses zum Anfangswert F_0. Da der mittlere Krümmungsfluss isotrop ist, ist das Verhalten von F_t durch das Verhalten von $c_t = M_t \cap \mathbb{R}^3_{>0}$ unter dem gestörten mittleren Krümmungsfluss

$$\frac{\partial}{\partial t} c = H + \lambda, \tag{0.1a}$$

$$c(\cdot, 0) = c_0, \quad c_t \subset \mathbb{R}^3_{>0}, \tag{0.1b}$$

bestimmt. Hierbei bezeichnen H den (mittleren) Krümmungsvektor der Kurve und λ einen Störterm erster Ordnung, der durch die Rotation entsteht. Weiter sei $[0, T)$, $T < \infty$, das maximale Zeitintervall, in dem (0.1) eine glatte Lösung besitzt.

Bei der Analyse des singulären Verhaltens der Kurven c unter dem gestörten Fluss (und damit der erzeugten Flächen M unter dem mittleren Kümmungsfluss) sind zwei Fälle zu unterscheiden. Zunächst werden Singularitäten untersucht, die entfernt von der Rotationsebene $\left\{ y \in \mathbb{R}^4, y^1 = 0, y^4 = 0 \right\}$ entstehen. In diesem Fall bleibt der Störterm λ auf dem gesamten Zeitintervall $[0, T)$ beschränkt und das singuläre Verhalten unter dem gestörten Fluss ist gleich dem singulären Verhalten von Kurven unter dem mittleren Krümmungsfluss. Das heißt, die Kurven werden in der Singularität eben in dem Sinne, dass das Verhältnis von Torsion zu Krümmung verschwindet: $\lim_{n\to\infty} \frac{|\tau(p_n, t_n)|}{|H(p_n, t_n)|} = 0$, wobei (p_n, t_n) eine essentielle Blow-up Folge bezeichnet. Folglich explodiert die Torsion in der Singularität nicht so schnell wie die Krümmung. Dies impliziert, dass nach einer entsprechenden Reskalierung entlang einer essentiellen Blow-up Folge (p_n, t_n) die Kurven entlang einer Teilfolge zu einer Familie ebener, konvexer Kurven konvergieren, die dem mittleren Krümmungsfluss genügen.

Als zweites werden die Singularitäten auf der Rotationsebene betrachtet, in diesem Fall explodiert der Störterm λ: $\overline{\lim}_{t\to T} |\lambda|^2 = \infty$. Zunächst wird eine Monotonieformel für den gestörten Fluss (0.1) bewiesen. Diese impliziert die Konvergenz der reskalierten Kurven zu einer selbstähnlichen Lösung des gestörten Flusses $c_\infty^\perp = -H_\infty - \lambda_\infty$ im Fall einer Typ-I Singularität in der sowohl die Krümmung der Kurve als auch der Störterm explodieren.

AMS Subject Classification: 53C44, 53A04, 53A07.

Schlagworte: mittlerer Krümmungsfluss, Rotationssymmetrie, curve shortening flow, höhere Kodimension

Abstract

In this thesis, mean curvature flow of rotationally symmetric surfaces immersed in $\mathbb{R}^4$ is studied. Let $c_0 : S^1 \to \mathbb{R}^3_{>0}$ be a closed immersed curve in the halfspace $\mathbb{R}^3_{>0} := \{y \in \mathbb{R}^3; y^1 > 0\}$ then rotation of c_0 in $\mathbb{R}^4$ defines an immersion $F_0 : S^1 \times S^1 \to \mathbb{R}^4$ such that the image M_0 is invariant under all rotations in the y^1y^4-plane:

$$M_0 := \left\{ \left(c^1\left(x^1\right)\cos x^2, c^2\left(x^1\right), c^3\left(x^1\right), c^1\left(x^1\right)\sin x^2\right); x^1, x^2 \in S^1 \right\}.$$

Let F_t be the one-parameter family of immersions evolving under mean curvature flow with initial data F_0 and $c_t := M_t \cap \mathbb{R}^4$. Since the mean curvature flow is isotropic, the behaviour of F_t is totally determined by the behaviour of the generating curves c_t evolving under the perturbed mean curvature flow

$$\frac{\partial}{\partial t}c = H + \lambda, \tag{0.1a}$$

$$c(\cdot, 0) = c_0, \quad c_t \subset \mathbb{R}^3_{>0}, \tag{0.1b}$$

where H is the (mean) curvature vector of the curve c_t and λ is a perturbation vector of first order coming from the rotation. Furthermore, $T < \infty$ denotes the maximal time up to which a smooth solution to (0.1) exists.

When analyzing the singular behaviour of (0.1) (and therefore the singular behaviour of the surfaces) one has to distinguish two cases. First, the singular behaviour is studied if the singularity occurs away from the plane of rotation $\{y \in \mathbb{R}^4, y^1 = 0, y^4 = 0\}$, i. e. if λ remains bounded. In this case, curves evolving under the perturbed flow behave like curves under mean curvature flow in the singularity: the curves become planar in the singularity in the sense that the ratio of the torsion and the curvature vanishes, hence, the torsion do not blow up as fast as the curvature $\lim_{n\to\infty} \frac{|\tau(p_n,t_n)|}{|H(p_n,t_n)|} = 0$ for an essential blow-up sequence (p_n, t_n). Furthermore, it is shown that after rescaling along an essential blow-up sequence (p_n, t_n) the solutions converge along a subsequence to a family of convex, planar curves evolving under mean curvature flow.

Secondly, the behaviour in a singularity on the plane of rotation, i. e. $\overline{\lim}_{t\to T} |\lambda|^2 = \infty$, is investigated. In this case, a monotonicity formula for the perturbed flow (0.1) is proved. If the occurring singularity is of type-I and if both the curvature vector of the curve and the perturbation term λ blow up, the monotonicity formula implies convergence of rescaled solutions to a self-similar solution of the perturbed flow $c_\infty^\perp = -H_\infty - \lambda_\infty$.

AMS Subject Classification: 53C44, 53A04, 53A07.

Key words and phrases: mean curvature flow, rotationally symmetric surfaces, curve shortening flow, higher codimension

Contents

1. Introduction

Let M^n be a smooth manifold immersed in a Riemannian manifold (N, h) locally given by $F_0 : U \subset \mathbb{R}^n \to F_0(U) \subset M_0$. A one-parameter family of immersions $F_t : M_t \to N$ evolves under the *mean curvature flow* if

$$\frac{\partial}{\partial t} F = \mathcal{H}, \tag{1.1a}$$

$$F(\cdot, 0) = F_0, \tag{1.1b}$$

where $\mathcal{H}$ is the mean curvature vector of M_t. Roughly speaking, the manifolds M_t change in time by moving each point in the direction of its mean curvature vector. It is well known (see for example [29]) that (1.1) has a solution for compact initial data M_0 at least for a small time $[0, T)$. The primary problem is to study the geometric properties of M_t in terms of M_0.

Geometrically, the mean curvature flow is the negative gradient flow of the volume functional, i. e. moving a manifold by its mean curvature vector is the fastest way to decrease its volume.

Furthermore, it is not hard to see, that if the initial manifold M_0 is compact and the ambient space is euclidean $F_0 : U \to F_0(U) \subset M_0 \subset \mathbb{R}^m$, the maximal time of existence is finite. The evolution equation of $|F|^2$

$$\frac{\partial}{\partial t} |F|^2 = \bar{\Delta} |F|^2 - 2n$$

and the parabolic maximum principle imply that $|F|^2 + 2nt$ is bounded from above for all $t \in [0, T)$ which contradicts $T = \infty$. Here $\bar{\Delta}$ is the Laplace-Beltrami operator on M and the Levi-Civita connection on M will be denoted by $\bar{\nabla}$ in the following.

In 1978, the mean curvature flow was studied by Brakke in [14], where he developed a general approach in arbitrary codimension via geometric measure theory and constructed a varifold solution for all time which has not to be unique. Further investigations on Brakke's flow especially on its singular behaviour were done by Ilmanen in [42] and White in [67].

Using parametric methods of differential geometry, mean curvature flow in codimension one has been studied extensively. At the end of the 1980s, Gage and Hamilton started analyzing the mean curvature flow for closed curves in euclidean space, the so called curve shortening flow. They proved that if the initial curve is planar, embedded and convex, the solution shrinks to a round point, i. e. after appropriate rescaling the curves converge smoothly to a round sphere. Grayson generalized this result to embedded plane curves in [31]. A different proof of his theorem was for example also given by Huisken in [40] and by Andrews and Bryan in [3]. Moreover, Grayson [32] and Gage [30] showed that an embedded curve on a surface converge to a point in finite time or to a geodesic in infinite time.

Sturmian oscillation theory for curve shortening was done by Angenent in 1991 ([7], [8]), he also studied the singular behaviour in [9]. In recent years, mean curvature flow for curves is still a matter of interest. In 2005 for example, Angenent proved the existence of geodesics of some prescribed topology on compact surfaces [11]. Curve shortening flow also initiated the study of generalized curve (shortening) flows (see for example [21], [35]).

Parallel to the progress in curve shortening flow, the first results for higher dimensional manifolds evolving under mean curvature flow were proved. In [36], 1984, Huisken showed that compact convex hypersurfaces of dimension $n \geqslant 2$ shrink to a round point in finite time. Even though the results for convex curves and convex hypersurfaces are similar, the methods used to prove them are much different. Huisken's proof uses the Codazzi-equations which are worthless in dimension one.

Ecker and Huisken also studied noncompact hypersurfaces evolving by mean curvature. In [23], they proved a maximum principle on noncompact manifolds and showed that if the initial data M_0 is an entire Lipschitz continuous graph and and grows at most linear, (1.1) has a smooth solution for all times. This result was generalized in [24].

Since mean curvature flow for codimension one is well understood, mean curvature flow for manifolds with higher codimension attracted more interest in recent years. Curve shortening flow for space curves was already studied in 1991. Altschuler and Grayson ([4], [5]) analyzed their singular behaviour and concluded that they become planar in the singularity. Their work initiated further investigation of curve shortening flow for example in euclidean manifolds of arbitrary dimension (cp. [44]) and also in higher dimensional non-euclidean manifolds (cf. [20]).

From the geometric point of view, a reason that the flow in higher codimension is much more complicated is that the normal bundle is no longer intrinsic in the sense that there is no canonical choice of a normal frame and therefore, the investigation of the second fundamental form become a delicate problem. One way of dealing with this is to equip the initial submanifold with some extra structure which has to be preserved under the mean curvature flow. For example, if M_0 is Lagrangian, the normal bundle can be identified with the tangent bundle and thus, it is intrinsic. Recall that an immersion $F_0 : M_0^n \to N$ in a Kähler manifold $\left(N^{2n}\omega, J, g\right)$ is called *Lagrangian* if the restriction of the sympletic form ω to the tangent bundle of M vanishes: $F^*\omega = 0$. Smoczyk proved in [54], 1996, that the mean curvature flow preserves the Lagrangian condition, if the ambient space is Kähler-Einstein. His result initiated the study of Lagrangian mean curvature flow. In [55], Smoczyk asked the question given a compact, oriented Lagrangian submanifold M immersed in a Calabi-Yau manifold which closed 1-forms m can be realized as the mean curvature form of another Lagrangian immersion of M into the same Calabi-Yau manifold. He also showed that a closed, oriented Lagrangian immersion which first cohomology class vanishes $[\mathcal{H}] = 0$ cannot develop a *type-I singularity* (for a definition cp. (1.3) below) under the mean curvature flow. Further investigations on singularities of Lagrangian mean curvature flow were done in [50], [51] and [33]. In [56], longtime existence and convergence of Lagrangian mean curvature flow in a flat space were considered. Smoczyk and Wang defined a generalized Lagrangian mean curvature flow for almost Lagrangian submanifolds in almost Kähler manifolds in [58] and proved that the flow has a solution at least for small times and it preserves the Lagrangian condition.

Wang [63] and Chen and Tian [16] proved that symplectic surfaces in Kähler-Einstein manifolds remain symplectic under the mean curvature flow. They also investigated the singular behaviour and proved that there cannot occur a type-I singularity (Wang [63], Chen and Li [18]). For an overview see for example [64]. Wang also analyzed mean curvature flow of graphs of maps $f : \Sigma_1 \to \Sigma_2$ and proved longtime existence and convergence under suitable curvature conditions for the manifolds $\Sigma_1,\ \Sigma_2$ (see [65]).

As mentioned above, a solution to the mean curvature flow with compact initial data in euclidean space only exists on a finite time interval $[0,T)$. Furthermore, the norm of the second fundamental form $\mathcal{A} = \bar{\nabla} dF$ has to blow up in the singularity: $\limsup_{t\to T} |\mathcal{A}|^2 = \infty$. For analyzing the singular behaviour it is necessary to know its blow-up rate. Huisken gave a lower bound in [38]:

$$\max_t |\mathcal{A}|^2 \geqslant \frac{C_0}{T-t} \tag{1.2}$$

for some constant C_0. A singularity is called *type-I*, if there exists a constant C_1 such that

$$\max_t |\mathcal{A}|^2 \leqslant \frac{C_1}{T-t}, \tag{1.3}$$

otherwise it is of *type-II*. One simple example for a type-I singularity is the shrinking sphere. An example for a type-II singularity was given by Angenent in [9]: consider an immersed curve in the euclidean plane with two loops and positive curvature, then the smaller loop pinches off in finite time (cp. Figure 1.1.).

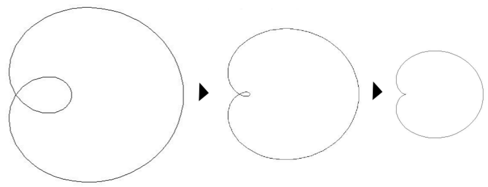

Figure 1.1.: Local convex curves may develop a type-II singularity

The monotonicity formula discovered by Huisken in [38] implies that each solution F to the mean curvature flow which develops a type-I singularity converge after homothetical rescaling to some nonempty limiting submanifold F_∞ satisfying

$$F_\infty^\perp = -\mathcal{H}_\infty, \tag{1.4}$$

i. e. F_∞ is a *self-shrinker*. Each solution F_0 of (1.4) defines a solution of the mean curvature flow moving by homotheties: $F(\cdot, t) := \sqrt{2(T-t)}F(\cdot, 0)$ satisfies

$$\left(\frac{\partial}{\partial t}F\right)^\perp = \mathcal{H}.$$

So, up to tangential deformation, F is a self similar shrinking solution to (1.1) and the classification of type-I singularities is equivalent to the classification of self-shrinkers. For curves $\gamma \subset \mathbb{R}^2$ this was done in 1986 by Abresch and Langer [1], and their result is valid for curves in arbitrary dimensional euclidean space, too. Huisken classified the compact hypersurfaces of dimension $n \geqslant 2$ with positive scalar mean curvature as spheres $S^m(\sqrt{m})$ of radius $\sqrt{m}$ in [38]. He also studied the noncompact hypersurfaces $M^m \subset \mathbb{R}^{m+1}$ which occur as type-I singularities of compact hypersurfaces with positive scalar mean curvature, [39]. Angenent gave examples of self-shrinkers with nonpositive scalar curvature in [10].

In higher codimension the problem is more delicate. In 2005, Smoczyk classified compact spherical self-shrinkers and noncompact self-shrinkers with uniformly bounded geometry in [57]. Furthermore, Joyce, Lee and Tsui constructed many self-similar solutions to the Lagrangian mean curvature flow in [45].

Another ansatz to handle singularities was developed by Evans, Spruck in [27] and by Chen, Giga, Goto in [15] independently from each other. In 1991, they defined the *level set flow* in which the moving surface is the zero-set of a function whose level-sets are moving by mean curvature. This approach gives weak solutions to the mean curvature flow, the so called viscosity solutions.

As mentioned above, mean curvature flow in higher codimension is not well understood, yet, and it is assumed that it is necessary to equip the initial manifold M_0 with extra structure to derive some results. In this thesis, the singular behaviour of rotationally symmetric surfaces immersed in euclidean space will be studied.

Let $c_0 : S^1 \to \mathbb{R}^3_{>0}$ be a closed curve in the halfspace $\mathbb{R}^3_{>0} := \left\{y \in \mathbb{R}^3; y^1 > 0\right\}$, then c_0 generates an immersion $F_0 : S^1 \times [0, 2\pi) \to \mathbb{R}^4$ such that its image M_0 is invariant under all rotations in the y^1y^4-plane:

$$M_0 := \left\{\left(c^1\left(x^1\right)\cos x^2, c^2\left(x^1\right), c^3\left(x^1\right), c^1\left(x^1\right)\sin x^2\right); x^1, x^2 \in S^1\right\}.$$

Rotationally symmetric hypersurfaces in euclidean space evolving under mean curvature flow have been studied extensively for compact initial data with boundary (e. g. [22]) and without boundary (e. g. [2], [43]). More generally, in [53] mean curvature flow for submanifolds which are invariant under the action of isometries of the ambient space were considered. Rotationally symmetric surfaces have been investigated also under the level set approach (e. g. [6], [59]).

Since the mean curvature flow is isotropic, the behaviour of F_t is totally determined by the behaviour of the generating curves c_t evolving under the perturbed mean curvature flow

$$\frac{\partial}{\partial t}c = H + \lambda, \tag{1.5a}$$

$$c(\cdot, 0) = c_0, \quad c_t \subset \mathbb{R}^3_{>0}, \tag{1.5b}$$

where H is the (mean) curvature vector of the curve c_t and λ is a perturbation vector of first order coming from the rotation.

Since λ is a term of first order, (1.5) has the same symbol as (1.1) and thus, there exists a solution on a maximal time interval. Moreover, each solution of (1.5) leads to a solution of (1.1) and vice versa, hence a solution of (1.5) exists as long as the corresponding solution of (1.1) exists. In Chapter 3, the perturbed flow for the generating curves will be described in more detail, and the evolution equations for the geometric quantities of interest will be calculated. Furthermore, some useful geometric properties for surfaces of codimension two will be given.

One problem appearing when studying curve shortening flow in higher codimension is, that contrary to curve shortening for plane curves there is no control of the curve's curvature from below. Even if initially the curvature vector H_0 does not vanish, there may occur inflection points $H = 0$ during the flow. In [4] this problem is solved via some dilation-invariant estimates which give control on the norm of the higher derivatives from above of the curvature vector on short time intervals. Because of the perturbation term λ in (1.5), these dilation-invariant estimates are not valid in general for the perturbed flow. In Chapter 2, the dilation-invariant estimates will be proved for the second fundamental form of an arbitrary dimensional manifolds evolving under mean curvature flow. Especially, these estimates are valid for the rotationally symmetric surface M. The proof is quite technical, but since these estimates are an important basis in the proof of the main theorems Theorem 1.1 and Theorem 1.2 below, it is given in detail.

The dilation-invariant estimates for the generated surface M will be used to get analogous estimates for the curve under the perturbed flow in Chapter 4. These estimates will give some control on the total loss of the curvature and the torsion on short intervals of time and space. This control and a further integral estimate will be needed to study the singular behaviour of c evolving under the perturbed flow, if the singularity occurs away from the plane of rotation $\left\{y \in \mathbb{R}^4, y^1 = 0, y^4 = 0\right\}$, i. e. if the perturbation vector λ remains bounded. In this case, λ should not be important in the singularity and in fact, it will be proved that the solutions to the perturbed flow behave like solutions to the curve shortening flow studied in [4], that is: in the singularity the solution become planar in the sense that the ratio of the torsion and the curvature vanishes, hence, the torsion do not blow up as fast as the curvature.

Theorem 1.1. *Assume that $|\lambda|^2 \leqslant \frac{1}{r_0^2}$ for some $r_0 > 0$ on $[0, T)$. If $\{(p_n, t_n)\}$ is an essential blow-up sequence, then*

$$\lim_{n \to \infty} \frac{|\tau(p_n, t_n)|}{|H(p_n, t_n)|} = 0. \tag{1.6}$$

After appropriate rescaling the solutions converge along a subsequence to some limit which is an ancient solution to the mean curvature flow. Furthermore, it follows from results of Altschuler in [4] that the limit is a family of convex planar curves. As a direct consequence, the generated surfaces converge:

Theorem 1.2. *Let $(p_n, 0, t_n) \in [0, 2\pi) \times [0, 2\pi) \times [0, T)$ be an essential blow-up sequence. Then there exists a subsequence of (p_n, t_n) along which the rescaled solutions of the generated surfaces M_n converge in $C^\infty(K)$ to a smooth, nontrivial limit M_∞, for every compact set $K \subset \mathbb{R}^2 \times (-\infty, 0]$. The limit M_∞ is a solution to the mean curvature flow and it exists at least on the time interval $(-\infty, 0]$.*

Moreover, M_∞ is a cylinder in $\mathbb{R}^4$ generated by c_∞ which is the limit of the rescaled curves along the subsequence, especially the mean curvature vector of the two limits M_∞ and c_∞ coincide $\mathcal{H}_\infty = H_\infty$. These theorems will be proved in Chapter 4, where also the definition of *essential blow-up sequence* will be given.

The investigation of singularities occurring on the plane of rotation is more delicate. In this case, a monotonicity formula for the perturbed flow will be proved in Chapter 5. If the curvature of the curve and the perturbation vector λ both blow up with the rate of a type-I singularity, a subsequence of a sequence of rescaled solutions will converge to a self-similar solution for the perturbed flow.

Theorem 1.3. *Assume that $\left(\frac{C_0}{T-t} \leqslant\right) \max_t \left(\frac{1}{r^2}\right) \leqslant \frac{C_1}{T-t}$ and $\left(\frac{C_2}{T-t} \leqslant\right) \max_t |H|^2 \leqslant \frac{C_3}{T-t}$ for some $C_0, \ldots, C_3 > 0$. Let p_0 be a blow-up point with $r_p = \langle e_1, p_0 \rangle = 0$ such that there exists a point p and a sequence $\{t_n\}$ with*

$$\lim_{n\to\infty} c(p, t_n) = p_0, \quad \lim_{n\to\infty} |H(p, t_n)|^2 = \infty, \quad \lim_{n\to\infty} |\lambda(p, t_n)|^2 = \infty,$$

i. e. the occurring singularity is of type-I and both, the curvature of the curve and the curvature coming from the rotation, blow up. Then the rescalings $\tilde{c}_n = \tilde{c}(\cdot, t_n)$ converge along a subsequence to some limit $\tilde{c}_\infty$ with $\tilde{c}_\infty^\perp = -\tilde{H}_\infty - \tilde{\lambda}_\infty$.

Since the study of the perturbed flow is just an auxiliary means, Theorem 1.3 is reformulated for the generated surfaces:

Corollary 1.4. *Let $F : S^1 \times S^1 \times [0, T) \to \mathbb{R}^4$ be a rotationally symmetric solution to the mean curvature flow such that the conditions of Theorem 1.3 are satisfied. Then the rescalings $\tilde{F}(\cdot, t_n)$ converge along a subsequence to some limit $\tilde{F}_\infty$ with $\tilde{F}_\infty^\perp = -\mathcal{H}$.*

The thesis ends with an appendix. In Chapter A, evolution equations for mean curvature flow of manifolds in arbitrary dimensional euclidean space are calculated in more details than in Chapter 2. Details omitted in Chapter 3 can be found in Chapter B. In Chapter C, the generalization of the results of Chapter 4 to arbitrary codimension will be discussed. An overview of the used symbols and abbreviations is given in the index.

There are many people who encouraged and supported me in the last years, thanks to all of them. Especially, I would like to thank my supervisor Prof. Dr. K. Smoczyk for his advice and guidance and for setting the main directions of research. I am grateful to Prof. Dr. H.-Ch. Grunau for acting as a referee. Thanks also to my colleagues from the Institute of Differential Geometry for the good co-operation and working atmosphere. I am thankful to Prof. Dr. J. Escher and Prof. Dr. G. Huisken for their helpful suggestions and advices.

2. Mean curvature flow in euclidean space

In this chapter, let $F_t : M_t^n \to \mathbb{R}^m$ be any family of immersed compact manifolds evolving under the mean curvature flow

$$\frac{\partial}{\partial t}F = \mathcal{H},$$
$$F(\cdot,0) = F_0,$$

where $\mathcal{H}$ is the mean curvature vector of the manifold M_t. One problem occurring when studying mean curvature flow in higher codimension is, that there is no control on the norm of the mean curvature vector from below. Even if initially $\mathcal{H} \neq 0$, the flow maybe develop inflection points $\mathcal{H} = 0$ during the flow. In [4], dilation-invariant estimates were proved for curves in $\mathbb{R}^3$ evolving under mean curvature flow. These estimates yield some control on the curvature of the curve on small intervals of time and space and ensure that the curvature do not vanish on these intervals. In the following, these estimates will be generalized to arbitrary dimensional manifolds. Even though the proof is quite technical, it is given in detail because of the vital importance of the estimates when analyzing the singular behaviour of rotationally symmetric surfaces M in Chapter 4. Before proving the estimates, the evolution equations of the higher covariant derivatives of the second fundamental will be given in the next section.

2.1. Evolution equations of the derivatives of the second fundamental form

In the following, local coordinates on M are denoted by $\left(x^i\right)_{i=1,\ldots,n}$, coordinates on $\mathbb{R}^m$ are denoted by $(y^\alpha)_{\alpha=1,\ldots,m}$. Doubled latin and greek indices are summed from 1 to n resp. from 1 to m. Let $\delta = \delta_{\alpha\beta}\, dy^\alpha \otimes dy^\beta$ be the euclidean metric on $\mathbb{R}^m$ then the coefficients of the induced metric on M are $\bar{g}_{ij} = \delta_{\alpha\beta}\, F_i^\alpha F_j^\beta$ with $F_i^\alpha := \frac{\partial F}{\partial x^i}$.

The Levi-Civita connection of M is denoted by $\bar{\nabla}$. The canonically induced connections on the pullback of the ambient space $F^{-1}T\mathbb{R}^{\mathrm{m}}$ or on the cotangent bundle T^*M of M are also denoted by $\bar{\nabla}$, $\bar{\Delta}$ is the Laplace-Beltrami operator.

The *second fundamental form* is defined by

$$\mathcal{A} := \bar{\nabla} dF =: \mathcal{A}_{ij}^\alpha \frac{\partial}{\partial y^\alpha} \otimes dx^i \otimes dx^j \text{ with } \mathcal{A}_{ij}^\alpha = F_{ij}^\alpha - F_m^\alpha \bar{\Gamma}_{ij}^m,$$

where $F_{ij}^{\alpha} := \frac{\partial^2 F^{\alpha}}{\partial x^i \partial x^j}$, and $\bar{\Gamma}_{ij}^k$ denote the Christoffel symbols on M. The Riemannian curvature on M is given by

$$\bar{R}\left(W, Z, X, Y\right) = \bar{g}\left(\bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z, W\right)$$

with local expression

$$\bar{R}_{klij} = \bar{R}\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \bar{g}_{kn}\left(\frac{\partial \bar{\Gamma}_{jl}^n}{\partial x^i} - \frac{\partial \bar{\Gamma}_{il}^n}{\partial x^j} + \bar{\Gamma}_{im}^n \bar{\Gamma}_{jl}^m - \bar{\Gamma}_{jm}^n \bar{\Gamma}_{il}^m\right).$$

The evolution equation of the second fundamental form can be deduced using Simons' identity (A.6), for detailed computations see Chapter A:

$$\frac{\partial}{\partial t}\mathcal{A}_{kl}^{\alpha} = \bar{\Delta}\mathcal{A}_{lk}^{\alpha} - 2\bar{g}^{ij}\mathcal{A}_{jm}^{\alpha}\bar{R}^m{}_{kli} - \mathcal{A}_{mk}^{\alpha}\bar{R}_l^m - \mathcal{A}_{lm}^{\alpha}\bar{R}_k^m + 2F_m^{\alpha}\mathcal{A}^{\alpha i m}\bar{\nabla}_i\mathcal{A}_{\alpha kl}. \tag{2.1}$$

Hence, the evolution equation of the norm is

$$\frac{\partial}{\partial t}\left|\mathcal{A}\right|^2 = \bar{\Delta}\left|\mathcal{A}\right|^2 - 2\left|\bar{\nabla}\mathcal{A}\right|^2 - 4\bar{g}^{ij}\mathcal{A}_{\alpha jm}\bar{R}^m{}_{kli}\mathcal{A}^{\alpha kl} - 4\mathcal{A}_{\alpha lm}\mathcal{A}^{\alpha lk}\bar{R}_k^m + 4\bar{a}^{ij}\bar{b}_{ij},$$

with $\bar{a}_{ij} = \delta_{\alpha\beta}\mathcal{A}_{ij}^{\alpha}\mathcal{H}^{\beta}$ and $\bar{b}_{ij} = \delta_{\alpha\beta}\bar{g}^{kl}\mathcal{A}_{ik}^{\alpha}\mathcal{A}_{jl}^{\beta}$. The Gauß equation (A.2) gives

$$\begin{aligned}\frac{\partial}{\partial t}\left|\mathcal{A}\right|^2 =& \bar{\Delta}\left|\mathcal{A}\right|^2 - 2\left|\bar{\nabla}\mathcal{A}\right|^2 - 4\bar{g}^{ij}\bar{g}^{mn}\mathcal{A}_{\alpha jm}\mathcal{A}^{\alpha kl}\mathcal{A}_{nl}^{\beta}\mathcal{A}_{\beta ki} \\ &+ 4\bar{g}^{ij}\bar{g}^{mn}\mathcal{A}_{\alpha jm}\mathcal{A}^{\alpha kl}\mathcal{A}_{ni}^{\beta}\mathcal{A}_{\beta kl} + 4\mathcal{A}^{\alpha i m}\mathcal{A}_{\alpha mj}\mathcal{A}^{\beta jn}\mathcal{A}_{\beta ni}.\end{aligned} \tag{2.2}$$

Comparing this equation to the evolution equation of the norm of the mean curvature of the manifold (cp. (A.12))

$$\frac{\partial}{\partial t}\left|\mathcal{H}\right|^2 = \Delta\left|\mathcal{H}\right|^2 - 2\left|\nabla\mathcal{H}\right|^2 + 4\bar{a}^{kl}\bar{a}_{kl} = \Delta\left|\mathcal{H}\right|^2 - 2\left|\nabla_i H^{\alpha} + \bar{a}_i^m F_m^{\alpha}\right|^2 + 2\bar{a}^{ij}\bar{a}_{ij}$$

illustrates why it is important to look at the second fundamental form instead of the mean curvature vector. In the proof of the dilation-invariant estimates (Lemma 2.4 below), the right hand sight of (2.2) will be roughly estimated in terms of $|\mathcal{A}|^2$, whereas the terms of the second fundamental form occurring on the right hand sight of (A.12) cannot be estimated in terms of the mean curvature vector.

For simplicity, the following abbreviation is used: If S and T are two tensors, $S * T$ denotes any linear combination of tensors formed by contraction on $S_{i,\ldots,j}T_{k\ldots,l}$ using the metric $\bar{g}^{ik}$.

Lemma 2.1. *For $N \in \mathbb{N}_0$ we denote the Nth iterated derivative of the the second fundamtental form by $\bar{\nabla}^N\mathcal{A} := \bar{\nabla}\ldots\bar{\nabla}\mathcal{A}$. The evolution equation of the Nth derivative $\bar{\nabla}^N\mathcal{A}$ is given by*

$$\begin{aligned}\frac{\partial}{\partial t}\left(\bar{\nabla}^N\mathcal{A}^{\alpha}\right) = \bar{\Delta}\left(\bar{\nabla}^N\mathcal{A}^{\alpha}\right) &+ \sum_{I+J+K=N}\bar{\nabla}^I\mathcal{A}^{\alpha} * \bar{\nabla}^J\mathcal{A}^{\beta} * \bar{\nabla}^K\mathcal{A}_{\beta} \\ &+ \sum_{\substack{I+J=N+1\\ I\leqslant N}} F_m^{\alpha}\bar{\nabla}^I\mathcal{A}^{\beta m} * \bar{\nabla}^J\mathcal{A}_{\beta}.\end{aligned} \tag{2.3}$$

Proof. For $N = 0$ this is (2.1). In the induction step $N \mapsto N+1$ the derivation in time is interchaged with the derivation in space. Hence, terms consisting of contractions of $\bar\nabla^N \mathcal{A}^\alpha$ and the time derivative of the Christoffel symbols occur. They can be expressed by contractions of the second fundamental form and its first covariant derivative (cf. (A.9)).

$$\begin{aligned}\frac{\partial}{\partial t}\left(\bar\nabla^{N+1}\mathcal{A}^\alpha\right) &= \bar\nabla\frac{\partial}{\partial t}\left(\bar\nabla^{N}\mathcal{A}^\alpha\right) + \bar\nabla^N\mathcal{A}^\alpha * \mathcal{A}^\beta * \bar\nabla\mathcal{A}_\beta\\ &= \bar\nabla\left(\bar\Delta\bar\nabla^N\mathcal{A}^\alpha + \sum_{I+J+K=N}\bar\nabla^I\mathcal{A}^\alpha * \bar\nabla^J\mathcal{A}^\beta * \bar\nabla^K\mathcal{A}_\beta\right.\\ &\qquad\left. + \sum_{\substack{I+J=N+1\\ I\leqslant N}} F^\alpha_m\bar\nabla^I\mathcal{A}^{\beta m} * \bar\nabla^J\mathcal{A}_\beta\right) + \bar\nabla^N\mathcal{A}^\alpha * \mathcal{A}^\beta * \bar\nabla\mathcal{A}_\beta\end{aligned}$$

Interchanging the covariant derivatives gives curvature terms which can also be expressed by contractions of the second fundamental form and its first covariant derivative.

$$\begin{aligned}\frac{\partial}{\partial t}\left(\bar\nabla^{N+1}\mathcal{A}^\alpha\right) =&\bar\Delta\bar\nabla^{N+1}\mathcal{A}^\alpha + \bar\nabla^N\mathcal{A}^\alpha * \bar\nabla\bar R + \bar\nabla^{N+1}\mathcal{A}^\alpha * \bar R\\ &+ \sum_{I+J+K=N+1}\bar\nabla^I\mathcal{A}^\alpha * \bar\nabla^J\mathcal{A}^\beta * \bar\nabla^K\mathcal{A}_\beta + \sum_{\substack{I+J=N+2\\ I\leqslant N+1}} F^\alpha_m\bar\nabla^I\mathcal{A}^{\beta m} * \bar\nabla^J\mathcal{A}_\beta\\ =&\bar\Delta\bar\nabla^{N+1}\mathcal{A}^\alpha + \sum_{I+J+K=N+1}\bar\nabla^I\mathcal{A}^\alpha * \bar\nabla^J\mathcal{A}^\beta * \bar\nabla^K\mathcal{A}_\beta\\ &+ \sum_{\substack{I+J=N+2\\ I\leqslant N+1}} F^\alpha_m\bar\nabla^I\mathcal{A}^{\beta m} * \bar\nabla^J\mathcal{A}_\beta.\end{aligned}$$

□

The evolution equations for the norm of all higher derivatives follow directly:

Corollary 2.2. *The evolution equation for the the norm of the Nth covariant derivative of $\mathcal{A}$ satisfies*

$$\begin{aligned}\frac{\partial}{\partial t}\left|\bar\nabla^N\mathcal{A}\right|^2 \leqslant&\bar\Delta\left|\bar\nabla^N\mathcal{A}\right|^2 - 2\left|\bar\nabla^{N+1}\mathcal{A}\right|^2 + d_{1,N}\left|\mathcal{A}\right|^2\left|\bar\nabla^N\mathcal{A}\right|^2\\ &+ 4d_{2,N+1}\left|\mathcal{A}\right|\left|\bar\nabla^{N+1}\mathcal{A}\right|\left|\bar\nabla^N\mathcal{A}\right| + d_{2,N}\left|\bar\nabla\mathcal{A}\right|\left|\bar\nabla^N\mathcal{A}\right|^2\\ &+ \sum_{1\leqslant I+J<N}\tilde d_{1,IJK}\left|\bar\nabla^I\mathcal{A}\right|\left|\bar\nabla^J\mathcal{A}\right|\left|\bar\nabla^{N-I-J}\mathcal{A}\right|\left|\bar\nabla^N\mathcal{A}\right|\\ &+ \sum_{1<I\leqslant N}\tilde d_{2,IJ}\left|\bar\nabla^I\mathcal{A}\right|\left|\bar\nabla^{N+1-I}\mathcal{A}\right|\left|\bar\nabla^N\mathcal{A}\right|,\end{aligned}$$

for constants $d_{1,IJK}, d_{2,IJ}$, etc.

2.2. Dilation-invariant estimates

In Chapter 4, the singular behaviour of M in the special case of rotationally symmetric surfaces will be studied. Therefore, it is necessary to control the total loss of the curvature of the generating curve. The dilation-invariant estimates in Lemma 2.4 will be used to get this control.

Definition 2.3. *The* maximum of the norm of the second fundamental form squared *at time t is denoted by* $\mathcal{M}_t := \max_{M_t} |\mathcal{A}|^2$.

Lemma 2.4. *For all $t_n \in [0, T)$, there exist constants $\sigma, \tilde{c}_N < \infty$ independent of t_n such that for $t \in \left(t_n, t_n + \frac{1}{2\sigma\mathcal{M}_{t_n}}\right]$ and $N \geqslant 0$ we have*

$$\left|\bar{\nabla}^N \mathcal{A}\right|^2 \leqslant \frac{\tilde{c}_N \mathcal{M}_{t_n}}{(t - t_n)^N}. \tag{2.4}$$

Indeed, σ does only depend on the dimension of M.

Here, dilation-invariance means (2.4) is invariant under rescaling of the form that time is scaled as space squared: $\tilde{F}(\cdot, \tau) := \mu F(\cdot, t)$, $\tau := \mu^2 t$ for some $\mu > 0$. This sort of rescaling will be used in Section 4.3 to show convergence along a subsequence of a blow-up sequence, cp. (4.23).

As mentioned above, these estimates were proved in [4, Theorem 3.1] for the curvature vector of space curves. For an arbitrary dimensional manifold M in euclidean space, the evolution equation of the squared norm of the second fundamental form is qualitative like the evolution equation of the the squared norm of the curvature vector of a space curve and so, the same methods as in [4] are applied in the following proof. It is quite technical, but because of the importance of Lemma 2.4 it is given in detail.

Proof of Lemma 2.4. The lemma is proved by induction on N. Without loss of generality, assume $t_n = 0$ and then translate the estimates. For $N = 0$, equation (2.2) implies that there exists a constant $\sigma > 0$ depending only on the dimension and codimension of M such that:

$$\frac{\partial}{\partial t} |\mathcal{A}|^2 \leqslant \bar{\Delta} |\mathcal{A}|^2 + \sigma |\mathcal{A}|^4 . \tag{2.5}$$

Set $h(\cdot, t) := |\mathcal{A}(\cdot, t)|^2 - \frac{\mathcal{M}_0}{1 - \sigma t \mathcal{M}_0}$, then (2.5) gives

$$\frac{\partial}{\partial t} h \leqslant \bar{\Delta} h + \sigma \left(|\mathcal{A}(\cdot, t)|^2 + \frac{\mathcal{M}_0}{1 - \sigma t \mathcal{M}_0} \right) h$$

and thus, the maximum principle (e. g. [52]) implies

$$\mathcal{M}_t \leqslant \frac{\mathcal{M}_0}{1 - \sigma t \mathcal{M}_0} \Longleftrightarrow -\frac{1}{\mathcal{M}_t} + \frac{1}{\mathcal{M}_0} \leqslant \sigma t \text{ if } t < \frac{1}{\sigma \mathcal{M}_0}. \tag{2.6}$$

This is

$$\mathcal{M}_t \leqslant 2\mathcal{M}_0 \text{ for } t \leqslant \frac{1}{2\sigma\mathcal{M}_0}$$

and we can choose $\tilde{c}_0 = 2$.

Furthermore, (2.6) leads to

$$-\frac{1}{\mathcal{M}_s} + \frac{1}{\mathcal{M}_t} \leqslant \sigma(s-t) \tag{2.7}$$

as long as $\mathcal{M}$ remains bounded, hence $\frac{1}{\mathcal{M}_t} \leqslant \sigma(T-t)$. Thus,

$$t_n + \frac{1}{2\sigma\mathcal{M}_{t_n}} \leqslant t_n + \frac{1}{2}(T-t_n) = \frac{1}{2}(T+t_n) \leqslant T,$$

i. e. $\left(t_n, t_n + \frac{1}{2\sigma\mathcal{M}_{t_n}}\right] \subset [0,T)$.

For $N=1$, Corollary 2.2 gives

$$\frac{\partial}{\partial t}\left|\bar{\nabla}\mathcal{A}\right|^2 \leqslant \bar{\Delta}\left|\bar{\nabla}\mathcal{A}\right|^2 - 2\left|\bar{\nabla}^2\mathcal{A}\right|^2 + d_1|\mathcal{A}|^2\left|\bar{\nabla}\mathcal{A}\right|^2 + 4d_2|\mathcal{A}|\left|\bar{\nabla}\mathcal{A}\right|\left|\bar{\nabla}^2\mathcal{A}\right| \tag{2.8}$$

for some constants d_1, d_2. The idea is, that the mixed term $4d_2|\mathcal{A}|\left|\bar{\nabla}\mathcal{A}\right|\left|\bar{\nabla}^2\mathcal{A}\right|$ can be estimated from above via $\mathsf{d}_1|\mathcal{A}|^2\left|\bar{\nabla}\mathcal{A}\right|^2 + \mathsf{d}_2\left|\bar{\nabla}^2\mathcal{A}\right|^2$ for some constants $\mathsf{d}_1, \mathsf{d}_2$ which can be chosen such that $\mathsf{d}_2\left|\bar{\nabla}^2\mathcal{A}\right|^2 - 2\left|\bar{\nabla}^2\mathcal{A}\right|^2 \leqslant 0$. So, the only problem is to control the first term $d_1|\mathcal{A}|^2\left|\bar{\nabla}\mathcal{A}\right|^2$. Since $t|\mathcal{A}|^2$ is bounded on small time intervals, compute $\frac{\partial}{\partial t}\left(t\left|\bar{\nabla}\mathcal{A}\right|^2 + d|\mathcal{A}|^2\right)$ and choose d large enough such that the negative gradient term occurring in the time derivative of $|\mathcal{A}|^2$ controls the term $t\mathsf{d}_1|\mathcal{A}|^2\left|\bar{\nabla}\mathcal{A}\right|^2$.

More precisely, note that $|\mathcal{A}|^2 \leqslant 2\mathcal{M}_0$ for $0 < t \leqslant \frac{1}{2\sigma\mathcal{M}_0}$, then (2.2) and (2.5) give

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(t\left|\bar{\nabla}\mathcal{A}\right|^2 + d|\mathcal{A}|^2\right) \\
&\quad\leqslant \bar{\Delta}\left(t\left|\bar{\nabla}\mathcal{A}\right|^2 + d|\mathcal{A}|^2\right) + \left|\bar{\nabla}\mathcal{A}\right|^2 - 2t\left(\left|\bar{\nabla}^2\mathcal{A}\right| - d_2|\mathcal{A}|\left|\bar{\nabla}\mathcal{A}\right|\right)^2 \\
&\qquad + \left(d_1 + 2d_2^2\right)t\left|\bar{\nabla}\mathcal{A}\right|^2|\mathcal{A}|^2 - 2d\left|\bar{\nabla}\mathcal{A}\right|^2 + c\cdot d|\mathcal{A}|^4 \\
&\quad\leqslant \bar{\Delta}\left(t\left|\bar{\nabla}\mathcal{A}\right|^2 + c|\mathcal{A}|^2\right) + \left(2\left(d_1 + 2d_2^2\right)t\mathcal{M}_0 - 2d + 1\right)\left|\bar{\nabla}\mathcal{A}\right|^2 + 4c\cdot d{\mathcal{M}_0}^2.
\end{aligned}$$

Choose d such that $\left(\frac{1}{\sigma}\left(d_1 + 2d_2^2\right) - 2d + 1\right) < 0$. For $t \leqslant \frac{1}{2\sigma\mathcal{M}_0}$ we have

$$\frac{\partial}{\partial t}\left(t\left|\bar{\nabla}\mathcal{A}\right|^2 + d|\mathcal{A}|^2\right) \leqslant \bar{\Delta}\left(t\left|\bar{\nabla}\mathcal{A}\right|^2 + d|\mathcal{A}|^2\right) + 4\sigma\cdot d{\mathcal{M}_0}^2.$$

Again, the maximum principle implies

$$t\left|\bar{\nabla}\mathcal{A}\right|^2 + d|\mathcal{A}|^2 \leqslant d\mathcal{M}_0 + 4\sigma\cdot d{\mathcal{M}_0}^2 t$$

which is $\left|\bar\nabla\mathcal{A}\right|^2 \leqslant \frac{3d\mathcal{M}_0}{t}$ for $t \leqslant \frac{1}{2\sigma\mathcal{M}_0}$. Note that d was chosen independent of t.

For $N > 1$, Corollary 2.2 yields:

$$\begin{aligned}\frac{\partial}{\partial t}\left(t^N\left|\bar\nabla^N\mathcal{A}\right|^2\right) \leqslant &\bar\Delta\left(t^N\left|\bar\nabla^N\mathcal{A}\right|^2\right) - 2t^N\left(\left|\bar\nabla^{N+1}\mathcal{A}\right| - d_{2,N+1}\left|\mathcal{A}\right|\left|\bar\nabla^N\mathcal{A}\right|\right)^2\\
&+ t^{N-1}\left(N + d_{2,N}t\left|\bar\nabla\mathcal{A}\right| + \left(2d_{2,N+1}^2 + d_{1,N}\right)t\left|\mathcal{A}\right|^2\right)\left|\bar\nabla^N\mathcal{A}\right|^2\\
&+ t^N\sum_{1\leqslant I+J<N} d_{1,IJK}\left|\bar\nabla^I\mathcal{A}\right|\left|\bar\nabla^J\mathcal{A}\right|\left|\bar\nabla^{N-I-J}\mathcal{A}\right|\left|\bar\nabla^N\mathcal{A}\right|\\
&+ t^N\sum_{1<I\leqslant N} d_{2,IJ}\left|\bar\nabla^I\mathcal{A}\right|\left|\bar\nabla^{N+1-I}\mathcal{A}\right|\left|\bar\nabla^N\mathcal{A}\right|\end{aligned}$$

As above, the terms of order N occurring in the sums must be controlled. Therefore, add a term $a_{N-1}t^{N-1}\left|\nabla^{N-1}\mathcal{A}\right|^2$ with a_{N-1} large enough such that the coefficient of the quadratic term of order N becomes negative:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(t^N\left|\bar\nabla^N\mathcal{A}\right|^2 + a_{N-1}t^{N-1}\left|\bar\nabla^{N-1}\mathcal{A}\right|^2\right) \leqslant \bar\Delta\left(t^N\left|\bar\nabla^N\mathcal{A}\right|^2 + a_{N-1}t^{N-1}\left|\bar\nabla^{N-1}\mathcal{A}\right|^2\right)\\
&+ t^{N-1}\underbrace{\left(N + d_{2,N}t\left|\bar\nabla\mathcal{A}\right| + \left(2d_{2,N+1}^2 + d_{1,N}\right)t\left|\mathcal{A}\right|^2 - a_{N-1}\right)}_{=:-d_1}\left|\bar\nabla^N\mathcal{A}\right|^2\\
&+ t^N\sum_{1\leqslant I+J<N} d_{1,IJK}\left|\bar\nabla^I\mathcal{A}\right|\left|\bar\nabla^J\mathcal{A}\right|\left|\bar\nabla^{N-I-J}\mathcal{A}\right|\left|\bar\nabla^N\mathcal{A}\right|\\
&+ t^N\sum_{1<I\leqslant N} d_{2,IJ}\left|\bar\nabla^I\mathcal{A}\right|\left|\bar\nabla^{N+1-I}\mathcal{A}\right|\left|\bar\nabla^N\mathcal{A}\right|\\
&+ a_{N-1}t^{N-1}d_0\left|\mathcal{A}\right|\left|\bar\nabla^{N-1}\mathcal{A}\right|\left|\bar\nabla^N\mathcal{A}\right| + a_{N-1}t^{N-1}\cdot(\text{terms of order } N-2)\,.\end{aligned}$$

Note that a_{N-1} was chosen independent of $0 < t \leqslant \frac{1}{2\sigma\mathcal{M}_0}$. Hence, the mixed terms in the sums are compensated:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(t^N\left|\bar\nabla^N\mathcal{A}\right|^2 + a_{N-1}t^{N-1}\left|\bar\nabla^{N-1}\mathcal{A}\right|^2\right) \leqslant \bar\Delta\left(t^N\left|\bar\nabla^N\mathcal{A}\right|^2 + a_{N-1}t^{N-1}\left|\bar\nabla^{N-1}\mathcal{A}\right|^2\right)\\
&- t^{N-1}\Bigg(d_1\left|\bar\nabla^N\mathcal{A}\right| - t\sum_{1\leqslant I+J<N} d_{1,IJK}\left|\bar\nabla^I\mathcal{A}\right|\left|\bar\nabla^J\mathcal{A}\right|\left|\bar\nabla^{N-I-J}\mathcal{A}\right|\\
&\qquad -d_2\left|\mathcal{A}\right|\left|\bar\nabla^{N-1}\mathcal{A}\right| - t\sum_{1<I\leqslant N} d_{2,IJ}\left|\bar\nabla^I\mathcal{A}\right|\left|\bar\nabla^{N+1-I}\mathcal{A}\right|\Bigg)^2\\
&+ t^{N-2}\left|\bar\nabla^{N-1}\mathcal{A}\right|^2\Big(d_3t^3\left|\mathcal{A}\right|^2\left|\bar\nabla\mathcal{A}\right|^2 + d_4t^3\left|\bar\nabla^2\mathcal{A}\right|^2 + d_5t^3\left|\mathcal{A}\right|\left|\bar\nabla\mathcal{A}\right|\left|\bar\nabla^2\mathcal{A}\right|\\
&\qquad +d_6t^2\left|\mathcal{A}\right|^2\left|\bar\nabla\mathcal{A}\right| + +d_7t^2\left|\mathcal{A}\right|\left|\bar\nabla^2\mathcal{A}\right|\Big)\\
&+ t^{N-1}\left|\bar\nabla^{N-1}\mathcal{A}\right|\cdot(\text{products of terms of lower order})\\
&+ a_{N-1}t^{N-1}\cdot(\text{terms of order } N-2)\,,\end{aligned}$$

for some constants d_i. Here, the coefficient of $t^{N-2}\left|\bar{\nabla}^{N-1}\mathcal{A}\right|^2$ is bounded independent of $0 < t \leqslant \frac{1}{2\sigma\mathcal{M}_0}$ by the induction hypothesis. So, there exist some constants a_I and A such that

$$\frac{\partial}{\partial t}\left(t^N\left|\bar{\nabla}^N\mathcal{A}\right|^2+\sum_{I=1}^{N-1}a_I t^I\left|\bar{\nabla}^I\mathcal{A}\right|^2\right)\leqslant\bar{\Delta}\left(t^N\left|\bar{\nabla}^N\mathcal{A}\right|^2+\sum_{I=1}^{N-1}a_I t^I\left|\bar{\nabla}^I\mathcal{A}\right|^2\right)+A{\mathcal{M}_0}^2.$$

Since $t^I\left|\bar{\nabla}^I\mathcal{A}\right|^2 \leqslant \tilde{C}_I\mathcal{M}_0$ for $0 < t \leqslant \frac{1}{2\sigma\mathcal{M}_0}$ the maximum principle yields the lemma. □

In [20] these estimates were proved for a family of curves evolving under mean curvature flow in a Riemannian manifold satisfying some curvature conditions.

Corollary 2.5. *For t_n large enough, there exists a constant $\sigma > 0$ such that $t_n = \tilde{t_n} + \frac{1}{4\sigma\mathcal{M}_{\tilde{t_n}}}$ for some earlier time $0 \leqslant \tilde{t_n} \leqslant t_n$.*

Proof. As above, the inequality $\frac{1}{\mathcal{M}_t} - \liminf_{s\to T}\frac{1}{\mathcal{M}_s} \leqslant \sigma(T-t)\,,\ t\in[0,T)\,,$ implies

$$t+\frac{1}{4\sigma\mathcal{M}_t}\leqslant t+\frac{1}{4}(T-t)\,.$$

Clearly, we get $t+\frac{1}{4\sigma\mathcal{M}_t} < t_n < t_n + \frac{1}{4\sigma\mathcal{M}_{t_n}}$ for $t < \frac{4}{3}t_n - \frac{1}{3}T < t_n$. So we can find $\tilde{t_n} \geqslant \frac{4}{3}t_n - \frac{1}{3}T$ such that $t_n = \tilde{t_n} + \frac{1}{4\sigma\mathcal{M}_{\tilde{t_n}}}$. □

3. Mean curvature flow for rotationally symmetric surfaces

From now on, the special case that M is a rotationally symmetric surfaces is considered. In this chapter, the setting will be described in detail not only for surfaces but more generally for higher dimensional manifolds always having in mind that we will restrict ourselves to surfaces generated by curves (and hence $n = 1$) below. First, the perturbed flow for the generating manifolds c will be derived, secondly, some evolution equations will be given. The chapter ends with some useful geometric properties for surfaces of codimension two.

Let $c_0 : [0, 2\pi)^n \to \mathbb{R}^m_{>0}$, $m \geqslant n + 2$, be a smooth immersed compact manifold in the halfspace $\mathbb{R}^m_{>0} := \left\{y \in \mathbb{R}^m ; y^1 > 0\right\} \subset \mathbb{R}^{m+1}$. As above, y^α, $\alpha = 1, \ldots, m + 1$, denote euclidean coordinates on $\mathbb{R}^{m+1}$. Let $F_0 : [0, 2\pi)^{n+1} \to M_0 \subset \mathbb{R}^{m+1}$ be the immersion generated by c_0 such that M_0 is invariant under all rotations in the $y^1 y^{m+1}$-plane:

$$\begin{aligned} M_0 &:= F_0 \left([0, 2\pi)^n \times [0, 2\pi)\right) \\ &:= \Big\{ \left(c^1 (\vec{x}) \cos x^{n+1}, c^2 (\vec{x}), \ldots, c^m (\vec{x}),\ c^1 (\vec{x}) \sin x^{n+1}\right) \in \mathbb{R}^{m+1}; \\ &\qquad\qquad\qquad\qquad\qquad \vec{x} \in [0, 2\pi)^n, x^{n+1} \in [0, 2\pi) \Big\} . \end{aligned}$$

From now on, $\bar{g} = \bar{g}_{ij}\, dx^i \otimes dx^j$ denotes the induced metric of the immersion F_t, resp. $g = g_{ij}\, dx^i \otimes dx^j$ the induced metric on c, always having in mind that the first sum goes from 1 to $n + 1$ and the second from 1 to n. As usual, the inverse of the metric is written as $\bar{g}^{ij}$ resp. g^{ij}. The euclidean metric on $\mathbb{R}^m$ will be denoted by $\delta_{\alpha\beta}$. For better reading, the scalar products and norms corresponding to these metrics and those which are induced by them are abbreviated by $\langle \cdot, \cdot \rangle$ resp. $|\cdot|$ if it is clear which metric is used from the context. Furthermore, $\bar{\nabla}$ (resp. ∇) denotes the Levi-Civita-connection on M (resp. c), $\bar{\Delta}$ (resp. Δ) is the Laplace-Beltrami-operator on M (resp. c).

For the rest of this thesis, the behaviour of M_0 under the mean curvature flow (mcf) will be investigated in the special case that M_0 is a compact surface: let $F_t : [0, 2\pi)^{n+1} \to \mathbb{R}^{m+1}$ be a family of smooth immersions such that

$$\frac{\partial}{\partial t} F\left(\vec{x}, x^{n+1}, t\right) = \mathcal{H} \tag{3.1a}$$

$$F\left(\vec{x}, x^{n+1}, 0\right) = F_0\left(\vec{x}, x^{n+1}\right), \tag{3.1b}$$

where $\mathcal{H}$ is the mean curvature vector of the immersion $F(\cdot, t)$. Below, the immersion F_t and its image M_t are often identified.

It is well known that (3.1) has a smooth solution on a maximal time interval $[0, T)$, see for example [29]. Since M is compact, the maximal time of existence is finite $T < \infty$. Furthermore, the mean curvature flow is isotropic, so M_t, $t \in [0, T)$, is rotationally symmetric, too, generated by

$$c_t(\vec{x}) := F_t\left(x^1, \dots, x^n, 0\right) = M_t \cap \mathbb{R}^m_{>0}.$$

Thus, the behaviour of M_t is totally determined by the behaviour of c_t.

To determine the flow under which the generating manifolds c_t evolve, choose cylindrical coordinates on $\mathbb{R}^{m+1}$ denoted by $\left(z^1, \dots, z^{m+1}\right) = \left(r, y^2, \dots, y^m, \phi\right)$ and compute the second fundamental form $\mathcal{A}$ of M in these coordinates. Details are given in Section B.1. The mean curvature vector $\mathcal{H}$ of M does not coincide with the mean curvature vector H of the generating manifold c, but (B.1) gives

$$\begin{aligned}
\mathcal{H}^\alpha\left(\vec{x}, x^{n+1}\right) &= \bar{g}^{ij}\mathcal{A}^\alpha_{ij}\left(\vec{x}, x^{n+1}\right) = g^{ij}A^\alpha_{ij}(\vec{x}) + \frac{1}{\left(c^1\right)^2}\mathcal{A}^\alpha_{n+1\,n+1}\left(\vec{x}, x^{n+1}\right)\\
&=: H^\alpha\left(\vec{x}, x^{n+1}\right) + \lambda^\alpha\left(\vec{x}, x^{n+1}\right) \text{ for } \alpha = 1, \dots, m,\\
\mathcal{H}^{m+1}\left(\vec{x}, x^{n+1}\right) &= 0,
\end{aligned}$$

where H is the mean curvature vector of c, $\lambda^\alpha := \frac{1}{(c^1)^2}\mathcal{A}^\alpha_{n+1\,n+1}$ denotes a vector due to the rotation:

$$\lambda^\alpha = -\frac{1}{c^1}\left(\delta^{1\alpha} - c^\alpha_m g^{ml} c^1_l\right) = -\frac{1}{c^1}\left(\frac{\partial}{\partial z^1} - \delta_{1\beta} c^\beta_m g^{ml} c^\alpha_l\right).$$

Set

$$e_1 := \frac{\partial}{\partial z^1} = \cos z^{m+1}\frac{\partial}{\partial y^1} + \sin z^{m+1}\frac{\partial}{\partial y^{m+1}}.$$

Since $\mathcal{H}$, λ do only depend on x^{n+1} as it describes the angle of rotation, the cut with the space $\left\{z^{m+1} = 0\right\}$ is permitted, and $\mathcal{H}(\vec{x}) = \mathcal{H}(\vec{x}, 0, t)$, $\lambda(\vec{x}) = \lambda(\vec{x}, 0, t)$ are identified with the corresponding vector fields in $\mathbb{R}^m$. Especially,

$$e_1\left(c(\vec{x}), 0\right) = \frac{\partial}{\partial y^1}\left(c(\vec{x})\right) \tag{3.2}$$

in euclidean coordinates on $\mathbb{R}^m$. Note that e_1 is constant with length

$$|e_1|^2 = \delta_{\alpha\beta}\, e^\alpha_1 e^\beta_1 = 1.$$

If the distance to the plane of rotation $\left\{y \in \mathbb{R}^m; y^1 = 0\right\}$ is denoted by

$$r(\vec{x}) = c^1(\vec{x}) = \langle c(\vec{x}), e_1\rangle \left(= \langle F(\vec{x}, 0), e_1\rangle\right), \tag{3.3}$$

then λ can be written as

$$\lambda = -\frac{1}{r}e^\perp_1, \tag{3.4}$$

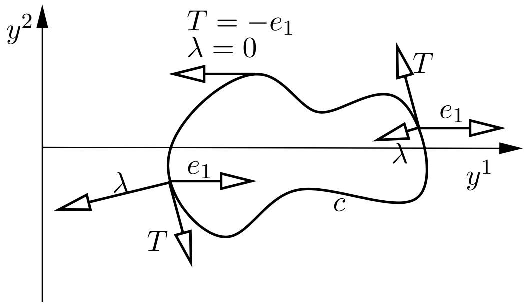

Figure 3.1.: Example of a plane curve

where, $^\perp$ is the orthogonal projection to the normal bundle NM of M (cp. Figure 3.1.).

Altogether, F_t is a solution to the mean curvature flow if and only if the generating immersion c_t are solutions to the perturbed flow

$$\frac{\partial}{\partial t} c\left(x,t\right) = H\left(x,t\right) + \lambda\left(x,t\right), \tag{3.5a}$$

$$c\left(x,0\right) = c_0\left(x\right), \; c\left(x,t\right) \in \mathbb{R}^m_{>0}. \tag{3.5b}$$

Since λ is a term of first order, (3.5) has the same symbol as (3.1) and thus, there exists a solution on a maximal time interval. Moreover, each solution of (3.5) leads to a solution of (3.1) and vice versa, hence a solution of (3.5) exists as long as the corresponding solution of (3.1) exists.

3.1. Evolution equations for space curves evolving under the perturbed flow

For the rest of this chapter, we look at the special case $n = 1$ and deduce some evolution equations for geometric quantities of the generating curve c evolving under the perturbed flow (3.5). Keep in mind that since c is a curve in $\mathbb{R}^m$, the latin indices only take the value $i = j = 1$ and repeated greek indices are summed from 1 to m.

Below, we will need the first two covariant derivatives of λ and the Laplacian of r.

Lemma 3.1. *For a curve c, we have*

$$\left|\nabla r\right|^2 = 1 - r^2\left|\lambda\right|^2, \tag{3.6a}$$

$$\Delta r = - rH^\alpha\lambda_\alpha, \tag{3.6b}$$

$$\nabla_l\lambda^\alpha = -\frac{1}{r}\frac{\partial r}{\partial x^l}\lambda^\alpha - H^\beta\lambda_\beta c^\alpha_l + \frac{1}{r}\nabla_l r H^\alpha \tag{3.6c}$$

$$\begin{aligned}\Delta\lambda^\alpha = &\left(\frac{2}{r^2}\left|\nabla r\right|^2 + H^\beta\lambda_\beta\right)\lambda^\alpha - 2\left(H^\beta\lambda_\beta + \frac{1}{r^2}\left|\nabla r\right|^2\right)H^\alpha + \frac{1}{r}\nabla^m r\nabla_m H^\alpha \\ &- \left(\nabla^m H^\beta\lambda_\beta - \frac{2}{r}\nabla^m r H^\beta\lambda_\beta + \frac{1}{r}\nabla^m r\left|H\right|^2\right)c^\alpha_m\end{aligned} \tag{3.6d}$$

Proof. The equations (3.6a)-(3.6c) are proved in Lemma B.1 for an arbitrary dimensional immersion c. Since c is a space curve, (B.9) implies (3.6d) via using $A_{ij}^{\alpha} = H^{\alpha} g_{ij}$. □

The key evolution equations are given in the following two lemmata:

Lemma 3.2. *For a family of space curves evolving under* (3.5) *we have*

$$\frac{\partial}{\partial t} r = \langle H + \lambda, e_1 \rangle = \Delta r - r\,|\lambda|^2\,, \tag{3.7}$$

$$\frac{\partial}{\partial t} c_i^{\alpha} = \nabla_i \left(H^{\alpha} + \lambda^{\alpha}\right) = \Delta c_i^{\alpha} + \nabla_i \lambda^{\alpha}, \tag{3.8}$$

$$\frac{\partial}{\partial t} g_{ij} = -2\left(|H|^2 + H^{\alpha}\lambda_{\alpha}\right) g_{ij}, \tag{3.9}$$

$$\frac{\partial}{\partial t}\left(d\mu\right) = -\left(|H|^2 + H^{\alpha}\lambda_{\alpha}\right) d\mu, \tag{3.10}$$

where $d\mu$ denotes the volume form of C.

These are standard computations as for example in [56], compare also Section A.2.

Lemma 3.3.

$$\begin{aligned}\frac{\partial}{\partial t} H^{\alpha} =& \Delta H^{\alpha} + \frac{1}{r}\nabla^m r \nabla_m H^{\alpha} + 2H^{\alpha}\left(|H|^2 - \frac{1}{r^2}\,|\nabla r|^2\right) + \lambda^{\alpha}\left(\frac{2}{r^2}\,|\nabla r|^2 + H^{\beta}\lambda_{\beta}\right)\\ &+ c_n^{\alpha}\left(\nabla^n\,|H|^2 - \nabla^n\,|\lambda|^2 + \frac{1}{r}\nabla^n r H^{\beta}\lambda_{\beta}\right)\end{aligned} \tag{3.11}$$

$$\begin{aligned}\frac{\partial}{\partial t}\,|H|^2 =& \Delta\,|H|^2 - 2\,|\nabla H|^2 + \frac{1}{r}\nabla^m r \nabla_m\,|H|^2\\ &+ 2H^{\alpha}\lambda_{\alpha}\left(\frac{2}{r^2}\,|\nabla r|^2 + H^{\beta}\lambda_{\beta}\right) + 4\,|H|^2\left(|H|^2 - \frac{1}{r^2}\,|\nabla r|^2\right).\end{aligned} \tag{3.12}$$

Proof. From (B.11), we get in the special case that C is a curve

$$\frac{\partial}{\partial t} H^{\alpha} = \Delta H^{\alpha} + \Delta\lambda^{\alpha} + c_n^{\alpha}\left(\nabla^n\,|H|^2 - \nabla^n\,|\lambda|^2 + \nabla^n\left(H^{\beta}\lambda_{\beta}\right)\right) + 2\left(|H|^2 + H^{\beta}\lambda_{\beta}\right) H^{\alpha}.$$

Hence, (3.6d) implies (3.11) and (3.12). □

3.2. Some geometric properties for codimension two surfaces

Now, assume $m = 3$, i. e. M is a surface in $\mathbb{R}^4$ generated by a curve $c \subset \mathbb{R}^3$. Since the only terms occurring are those with $i = j = 1$, the local coordinate of c will be denoted by x instead of x^1.

Suppose that $|H(x,t)| > 0$ for all $(x,t) \in S^1 \times [0,T)$. The *principal unit normal vector field* of c is

$$\nu = \frac{1}{|H|} H.$$

Let T be a unit tangent vector field representing the orientation of c: $T = \frac{1}{v}\frac{\partial c}{\partial x}$, $v := \left|\frac{\partial c}{\partial x}\right|$. Then there exists a uniquely defined unit normal vector field B such that T, ν, B is an positive oriented orthonormal frame field of $\mathbb{R}^3$. B is called the *binormal vector field* of c. Then the *Frenet equations* are given by

$$A_{ij}^{\alpha} = \nabla_i c_j^{\alpha} = |H|\, g_{ij}\, \nu^{\alpha} \tag{3.13a}$$
$$\nabla_i \nu^{\alpha} = -|H|\, c_i^{\alpha} + \tau_i B^{\alpha} \tag{3.13b}$$
$$\nabla_i B^{\alpha} = -\tau_i \nu^{\alpha}. \tag{3.13c}$$

As computed in Section B.1, the second fundamental form on M is given in cylindrical coordinates by:

$$(\mathcal{A}_{kl}^{\alpha})_{k,l} = \begin{pmatrix} \frac{1}{|\dot{c}|^2} H^{\alpha} & 0 \\ 0 & \frac{1}{(c^1)^2}\lambda^{\alpha} \end{pmatrix}, \text{ for } \alpha = 1, \dots, 3, \quad \mathcal{A}_{kl}^{4} = 0,$$

where the first coordinate represent the radius, the second one denotes the angle of rotation. This gives

$$|\mathcal{A}|^2 = |H|^2 + |\lambda|^2 \tag{3.14a}$$
$$|\mathcal{H}|^2 = |H|^2 + 2\langle H, \lambda\rangle + |\lambda|^2 \leqslant 2|\mathcal{A}|^2. \tag{3.14b}$$

Since M is a surface, the Gaussian curvature computes to

$$2K = |\mathcal{H}|^2 - |\mathcal{A}|^2 = 2\langle H, \lambda\rangle.$$

Hence the theorem of Gauß-Bonnet implies

$$\int_M \langle H, \lambda\rangle\, \mathrm{d}\eta = \int_C r\,\langle H, \lambda\rangle\, \mathrm{d}\mu = 0.$$

This is also a direct consequence of (3.6b) and the divergence theorem.

Definition 3.4. *The* tangent indicatrix *of c is the curve on the unit sphere given by $\gamma(s) = T$.*

In general the indicatrix to a given curve c is not parametrized by arc length, but for the arc length parameter s of c we have

$$\frac{\partial \gamma}{\partial s} = \frac{\partial^2 c}{\partial s^2} = H = |H|\,\nu.$$

If the curvature $|H|$ does not vanish on c,

$$\frac{\partial}{\partial\sigma} := \frac{1}{|H|}\frac{\partial}{\partial s}, \quad d\sigma = |H|\, ds$$

is well defined and

$$\frac{\partial\gamma}{\partial\sigma} = N, \quad \frac{\partial^2\gamma}{\partial\sigma^2} = -T + \frac{\tau}{|H|}B.$$

So the torsion of the indicatrix as a space curve is $\frac{\tau}{|H|}$. A well known fact from the theory of curves says that a regular parametrized space curve with non vanishing curvature is planar on a subarc if and only if the torsion vanishes on the subarc, see Lemma 3.6. Hence, if the indicatrix lies on a great circle of the sphere along a subarc (or equivalently: if c is planar along this subarc), and if $H \neq 0$, $\frac{\tau}{|H|}$ has to vanish. Therefore, the following definition is proper.

Definition 3.5. *A space curve c is called* planar at a point p *if* $\frac{\tau}{|H|}(p) = 0$.

In general, if c is a curve in arbitrary dimensional euclidean space, one can define $m-2$ different torsion 1-forms τ_i (cp. (C.1)) and the fact mentioned above can be proved for arbitrary codimension, see for example [61, Theorem 5, p. 27 f.]:

Lemma 3.6. *Let $c : [a, b] \to \mathbb{R}^m$ be a regularly parametrized curve with $|H|, \tau_1, \ldots, \tau_{j-1}$ nowhere zero, and τ_j everywhere zero. Then c lies in some $(j+1)$-dimensional plane in $\mathbb{R}^n$.*

Moreover, in Chapter C, Lemma 3.6 will be used to generalize the results of the following chapter for surfaces in $\mathbb{R}^4$ to surfaces in $\mathbb{R}^m$.

4. Formation of singularities away from the plane of rotation

Consider a family of surfaces M generated by space curves c, where $c : S^1 \times [0,T) \to \mathbb{R}^3_{>0}$ is a solution to the perturbed mean curvature flow

$$\frac{\partial}{\partial t} c\,(x,t) = H\,(x,t) + \lambda\,(x,t)\,, \tag{4.1a}$$

$$c\,(x,0) = c_0\,(x)\,,\;\; c\,(x,t) \in \mathbb{R}^3_{>0} \tag{4.1b}$$

on the maximal time interval of existence $[0,T)$. In this chapter, singularities of the flow occurring away from the plane of rotation will be analyzed. From now on, assume that

$$\textbf{There exists a constant } r_0 > 0 \textbf{ such that } r \geqslant r_0 \textbf{ for all } 0 \leqslant t < T. \tag{4.2}$$

is satisfied.

The following lemma ensures that assumption (4.2) is not empty, in fact, there are a lot of curves for which (4.2) is fulfilled.

Lemma 4.1. *Let $c : S^1 \times [0,T) \to \mathbb{R}^3_{\geqslant 0}$ be a solution to the perturbed flow (4.1). Assume that c_0 is contained in a closed ball $\overline{B\,(\rho, M)} := \left\{ y \in \mathbb{R}^3; |y - M|^2 \leqslant \rho^2 \right\}$ of radius ρ and centre M with $\langle M, e_1 \rangle = \varepsilon_0 + \rho$ for some $\varepsilon_0 > 0$ (see Figure 4.1.).*

If $\rho < \varepsilon_0$, the solution is a subset of $\overline{B\,(\rho, M)}$ for all $t \in [0,T)$, hence, $r \geqslant \varepsilon_0 > 0$ for all $t \in [0,T)$, i. e. assumption (4.2) is satified with $r_0 = \varepsilon_0$.

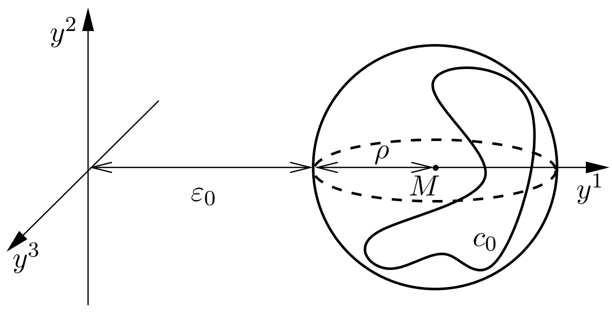

Figure 4.1.: Conditions in Lemma 4.1

Proof. Let $c \subset \overline{B\,(\rho, M)}$ be a solution to the perturbed flow with $|c - M|^2 \leqslant \rho^2$. Since $\max_t |c - M|^2$ is continuous, assume that $\max_{t_0} |c - M|^2 = \rho^2$ at a first time t_0. Then, there exists a point x_0 such that $\max_{t_0} |c - M|^2 = |c - M|^2 (x_0, t_0)$, and $c(\cdot, t_0)$ touches $S^2(\rho, M)$ in x_0 tangentially. At this point, we have

$$\Delta|_{x=x_0} |c - M|^2 = 2 \left(\langle H, c - M \rangle + 1\right)|_{x=x_0} \leqslant 0.$$

Since $c(x_0, t_0) - M$ is orthogonal to the sphere and hence to c itself in x_0, the Cauchy-Schwarz inequality yields

$$\langle \lambda, c - M \rangle|_{x=x_0} = -\frac{1}{r} \langle e_1, c - M \rangle \bigg|_{x=x_0} \leqslant \frac{\rho}{r(x_0, t_0)} \leqslant \frac{\rho}{\varepsilon_0},$$

where $r(x_0, t_0) \geqslant \varepsilon_0$ was used due to the condition on the centre M of the ball. Altogether, the assumption $\rho < r_0$ implies

$$\frac{\partial}{\partial t}\bigg|_{t=t_0} \left(|c - M|^2\right)(x_0) = 2 \langle H + \lambda, c - M \rangle|_{x=x_0} \leqslant 2 \left(\frac{\rho}{\varepsilon_0} - 1\right) < 0,$$

which is a contradiction to $\frac{\partial}{\partial t}\big|_{t=t_0} \left(|c - M|^2 (x_0)\right) \geqslant 0$ by the choice of t_0. In the worst case, every inequality is an equality and $r(x_0, t_0) = \varepsilon_0$, hence, the condition cannot be sharpened. □

Roughly, the idea of the previous lemma is that the curvature of the barrier $S^2(\rho, M)$ constrains a lower bound on the curvature of the curve in a point in which the curve first touches the sphere. If this lower bound is good enough, i. e.

$$|H(x_0, t_0)| \geqslant \frac{1}{\rho} > \frac{1}{\varepsilon_0} \geqslant |\lambda(x_0, t_0)|,$$

the mean curvature vector forces the curve back inside the ball.

But the condition $\varepsilon_0 < \rho$ is also necessary to ensure that a solution to the perturbed flow with initial data $c_0 \subset S^2(\rho, M)$ develops a singularity away from the plane of rotation. In the following example, the flow in direction of the mean curvature vector does not influence the flow in the direction of λ, i. e. e_1 is everywhere perpendicular to c.

Example Let c_0 be a circle in a affine y^2y^3-plane, i. e. (cp. Figure 4.2.)

$$c_0 : S^1 \to \mathbb{R}^3_{\geqslant 0}, \quad c_0(x) = (r, \rho \cos x, \rho \sin x), \quad r, \rho > 0 \text{ constant}.$$

In this case $\lambda = -\frac{1}{r} e_1^\perp = -\frac{1}{r} e_1$, and c is a solution to the perturbed flow, if r and ρ are a solutions to $\frac{\partial}{\partial t} r(t) = -\frac{1}{r(t)}$, $\frac{\partial}{\partial t} \rho(t) = -\frac{1}{\rho(t)}$. This is

$$c(x, t) = \left(\sqrt{r^2 - 2t}, \sqrt{\rho^2 - 2t} \cos x, \sqrt{\rho^2 - 2t} \sin x\right)$$

and the solution exists for $t < \frac{1}{2} \min(r, \rho)$. Hence, the solution shrinks to a point not lying in the plane of rotation if and only if $\rho < r$.

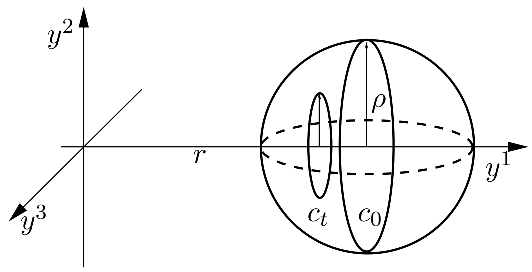

Figure 4.2.: Circles in an affine y^2y^3-plane remain circles under the perturbed flow.

Moreover, the surface $(x,t) \mapsto c(x,t)$ satisfies:

$$-\left(c_t^1(x)\right)^2 + \left(c_t^2(x)\right)^2 + \left(c_t^2(x)\right)^2 = \rho^2 - r^2 \begin{cases} < 0 & \text{if } r > \rho \\ = 0 & \text{if } r = \rho \\ > 0 & \text{if } r < \rho. \end{cases}$$

That means (cp. Figure 4.3.)

$$(x,t) \mapsto c(x,t) \text{ is } \begin{cases} \text{one sheet of a hyperboloid of two sheets, if } r > \rho \\ \text{a cone, if } r = \rho \\ \text{one half of a hyperboloid of one sheet, if } r < \rho. \end{cases}$$

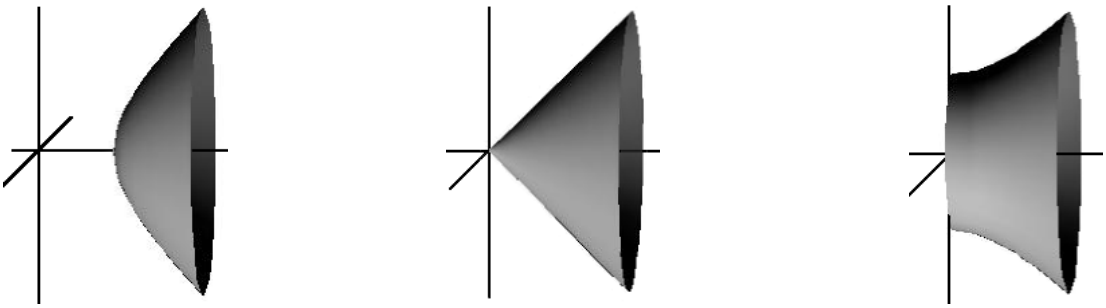

Figure 4.3.: The three cases of singular behaviour: λ remains bounded and H blows up, λ and H blow up, λ blows up and H remains bounded

The previous example shows:

Remark 4.2. The perturbed mean curvature flow (4.1) can develop three kinds of singularities: λ remains bounded and H blows up, λ and H blow up, λ blows up and H remains bounded.

Furthermore, the example illustrates that if the initial curve c_0 lies in a affine y^2y^3-plane, i. e. in a plane E_0 which is normal to e_1, the perturbed flow for the curves splits into two parts: the perturbation term λ leads only to a change of the affine plane $c_t \subset E_t$, $e_1 \perp E_t$, and do not influence the flow of the curves in the planes. Consequently, the perturbed flow for c_0 is similar to the planar curve shortening flow (see e. g. [29], [31]) as long as λ remains bounded.

If the initial data c_0 lies in a plane which contains e_1 and hance the perturbation term λ, the curves will not change the plane during the flow. This case was studied for example in [2] and [43].

4.1. Dilation-invariant estimates for the perturbed flow

In Subsection 4.1.1, the dilation-invariant estimates derived in Lemma 2.4 will be used to control the dissipation of the curvature and the torsion of the curve in small intervals of time and space. Below, these estimates are given in terms of the curves evolving under the perturbed flow.

In the following, we will often refer to the generated manifolds M moving by the mean curvature flow. Therefore, it is used that a parametrization x of the curves c gives a parametrization in cylindrical coordinates (x, ϕ) of M, where ϕ is the angle describing the rotation. Since M is rotationally symmetric, the angle ϕ is omitted where it is not necessary and the occurring terms are evaluated at the corresponding point with $\phi = 0$, for example $|\mathcal{A}(x)|^2 = |\mathcal{A}(x,0)|^2 = |\mathcal{A}(x,\phi)|^2$ for all ϕ.

As in [4], *(essential) blow-up sequences* will be used to analyze the singular behaviour of the perturbed flow:

Definition 4.3. *1. The maximum of the squared norm of second fundamental form of M will be denoted by $\mathcal{M}_t := \max_{M_t} |\mathcal{A}|^2$.*

2. The maximum of the curvature squared of c will be denoted by $\mathcal{N}_t := \max_{c_t} |H|^2$.

3. $\{(p_n, t_n)\}$, $p_n \in S^1$, $0 \leqslant t_n < T$, is a blow-up sequence *for $\mathcal{A}$ (resp. for H) if*

$$\lim_{n\to\infty} t_n = T \text{ and } \lim_{n\to\infty} |\mathcal{A}(p_n,t_n)|^2 = \infty$$

(resp. for H).

4. $\{(p_n, t_n)\}$, $p_n \in S^1$, $0 \leqslant t_n < T$, is an essential blow-up sequence *for $\mathcal{A}$ (resp. for H) if $\{(p_n, t_n)\}$ is a blow-up sequence and there exists a constant $\tilde{\rho} > 0$, independent of n, such that $\tilde{\rho}\mathcal{M}_t \leqslant |\mathcal{A}(p_n,t_n)|^2$ for all $0 \leqslant t \leqslant t_n$ (resp. for H).*

Remark 4.4. Condition (4.2) implies that λ is bounded from above $|\lambda|^2 \leqslant \frac{1}{r_0^2}$ for all time $0 \leqslant t < T$. Hence, the unboundedness of

$$|\mathcal{A}(p_n,t_n)|^2 = |H(p_n,t_n)|^2 + |\lambda(p_n,t_n)|^2$$

(cp. (3.14a)) along a blow up sequence (p_n, t_n) for $\mathcal{A}$ implies that $|H(p_n,t_n)|^2$ is also unbounded. Thus, (p_n, t_n) is a blow-up sequence for H, too. After introducing a subsequence n_k, if necessary, an essential blow-up sequence for $\mathcal{A}$ is also an essential blow-up sequence for H: If $\tilde{\rho}\mathcal{M}_t \leqslant |\mathcal{A}(p_n,t_n)|^2$ for all $0 \leqslant t \leqslant t_n$ then there exists a ρ such that $\rho\mathcal{N}_t \leqslant |A(p_{n_k},t_{n_k})|^2$ for $0 \leqslant t \leqslant t_n$. Conversely, every (essential) blow-up sequence for H admits an (essential) blow-up sequence for $\mathcal{A}$.

Consequently, an essential blow-up sequence (p_n, t_n) will always be chosen such that it is an essential blow-up sequence for both $\mathcal{A}$ and H. Furthermore, it is no restriction to assume $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$ for all t_n.

Note also that an essential blow-up sequence always exists. For example, choose a monotone increasing sequence s_n with $s_n \to T$ and set (p_n,t_n) such that $|H(p_n,t_n)|^2 = \max_{S^1\times[0,s_n]} |H|^2$. Then (p_n,t_n) is an essential blow-up sequence with $\rho = 1$.

Proposition 4.5. *There exist constants $\sigma, c_N < \infty$ such that for every time t_n we have:*

$$\left|\nabla^N H\right|^2 \leqslant c_N \frac{\mathcal{M}_{t_n}}{(t-t_n)^N}, \ \textit{for } t \in \left(t_n, t_n + \frac{1}{2\sigma\mathcal{M}_{t_n}}\right]. \tag{4.3}$$

Proof. From the explicit computation of $\mathcal{A}$ in cylindrical coordinates, one gets (cf. (B.4))

$$|\mathcal{A}|^2 = |H|^2 + |\lambda|^2, \quad \left|\nabla^N H\right|^2 = \left|\nabla^N A\right|^2 \leqslant \left|\bar{\nabla}^N \mathcal{A}\right|^2$$

Hence, the claim follows directly from Lemma 2.4 applied to the surface M generated by c. □

Particularly, in terms of the generating curves c this is:

Corollary 4.6. *Let $r \geqslant r_0$ for all times $t \in [0,T)$. There exist constants $\sigma, c_N < \infty$ such that for every time t_n with $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$ we have:*

$$\left|\nabla^N H\right|^2 \leqslant c_N \frac{\mathcal{N}_{t_n}}{(t-t_n)^N}, \ \textit{for } t \in \left(t_n, t_n + \frac{1}{4\sigma\mathcal{N}_{t_n}}\right], N \geqslant 0. \tag{4.4}$$

As in [4], these estimates also prove long time existence of the solution as long as the curvature and the perturbation term $|\lambda|^2$ remain bounded independent of the curve's torsion:

Proposition 4.7. *If $|H|$ and $|\lambda|$ are bounded on the time interval $[0,\omega)$, then there exists an $\varepsilon > 0$ such that $c(\cdot,t)$ exists and is smooth on the extended time interval $[0,\omega+\varepsilon)$.*

Proof. Under assumption (4.2) all derivatives of the tangent vector are bounded as long as the curvature remains bounded on the time interval $[0,\omega)$. Therefore, one can define a smooth limit of the tangent vector at time ω. Integration gives a smooth limit curve. By short time existence for the mean curvature flow of the generated manifold M one can extend the solution for a further time. □

Now, following the argumentation in [4], these estimates will be used to give some control on the loss of the curvature and torsion. Away from where the curvature vanishes the Frenet equations (3.13) imply:

$$
\begin{aligned}
&\left|\nabla_i c_j^\alpha\right|^2 = |H|^2\,,\\
&|\nabla_k \nabla_l c_i^\alpha|^2 = |\nabla |H||^2 + |H|^4 + |H|^2 |\tau_k|^2\,,\\
&|\nabla_j \nabla_i \nabla_k c_l^\alpha|^2\\
&\qquad = \left|\nabla_j \nabla_i |H| - |H|^3 - |H| \tau_i \tau_j\right|^2 + \left|3 |H| \nabla_j |H|\right|^2 + \left|2\nabla_j |H| \tau_i + |H| \nabla_j \tau_i\right|^2,\\
&|\nabla_m \nabla_j \nabla_i \nabla_k c_l^\alpha|^2\\
&\qquad = \left|3\nabla_m \nabla_j |H| \tau_i + 3\nabla_j |H| \nabla_m \tau_i + |H| \nabla_j \nabla_m \tau_i - |H|^3 g_{ij} \tau_m - |H| \tau_m \tau_i \tau_j\right|^2\\
&\qquad\quad + \left(\text{terms involving } \tau, |H|, \nabla |H|, \nabla \tau, \nabla^2 |H|, \nabla^3 |H|\right)^2.
\end{aligned}
$$

Proposition 4.5 gives bounds on the summands only in terms of the maximum of the second fundamental form of the generated surface:

Corollary 4.8. *Fix t_n large enough such that $t_n = \tilde{t_n} + \frac{1}{4\sigma \mathcal{M}_{\tilde{t_n}}}$ for some $\tilde{t_n} \in [0, t_n)$ and assume that $\rho \mathcal{M}_t \leqslant \mathcal{M}_{t_n}$ for all $t \leqslant t_n$. Then there exist constants $\tilde{c}_1, \ldots, \tilde{c}_4 < \infty$, depending only on ρ, such that for $t \in \left[t_n, t_n + \frac{\rho}{4\sigma \mathcal{M}_{t_n}}\right]$ the following bounds are valid*

$$
\begin{aligned}
&|H|^2 \leqslant \tilde{c}_1 \mathcal{M}_{t_n}, \quad |\nabla |H||^2 \leqslant \tilde{c}_2 {\mathcal{M}_{t_n}}^2, \quad |H|^2 |\tau|^2 \leqslant \tilde{c}_2 {\mathcal{M}_{t_n}}^2,\\
&|2\nabla_j |H| \tau_i + |H| \nabla_j \tau_i|^2 \leqslant \tilde{c}_3 {\mathcal{M}_{t_n}}^3, \quad \left|\nabla_j \nabla_i |H| - |H|^3 g_{ij} - |H| \tau_i \tau_j\right|^2 \leqslant \tilde{c}_3 {\mathcal{M}_{t_n}}^3,\\
&\left|3\nabla_m \nabla_j |H| \tau_i + 3\nabla_j |H| \nabla_m \tau_i + |H| \nabla_j \nabla_m \tau_i - |H|^3 g_{ij} \tau_m - |H| \tau_m \tau_i \tau_j\right|^2 \leqslant \tilde{c}_4 {\mathcal{M}_{t_n}}^4.
\end{aligned}
$$

If $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$, these bounds can be rewritten in term of $\mathcal{N}_{t_n}$.

Proof. For $\tilde{t_n}$ choose t_n as above (cf. Corollary 2.5), define the time interval $I_n := \left[t_n, t_n + \frac{\rho}{4\sigma \mathcal{M}_{t_n}}\right]$. Since $\mathcal{M}_{t_n} \geqslant \rho \mathcal{M}_{\tilde{t_n}}$, I_n is a subset of $\left[\tilde{t_n}, \tilde{t_n} + \frac{1}{2\sigma \mathcal{M}_{\tilde{t_n}}}\right]$. In this time interval the bounds the higher derivatives of H are valid (cf. Proposition 4.5) and the result follows easily:

$$|\nabla |H||^2 \leqslant |\nabla A|^2 \leqslant \frac{c\mathcal{M}_{\tilde{t_n}}}{t - \tilde{t_n}} \leqslant c\mathcal{M}_{\tilde{t_n}} \cdot 4\sigma \mathcal{M}_{\tilde{t_n}} \leqslant c_2(\rho) {\mathcal{M}_{t_n}}^2.$$

□

Below, we will see that the importance of these estimates lies in the fact that the bounds on the right hand side are given not in the earlier time $\tilde{t_n}$ but at time t_n.

4.1.1. Controlling the dissipation of the curvature and torsion

Away from where the curvature vanishes we can use the Frenet equations (3.13) and (3.12) to derive the evolution equation for the curvature $|H|$:

$$\begin{aligned}
2|H|\frac{\partial}{\partial t}|H| =& \Delta |H|^2 + 2|\nabla |H||^2 - 2|\nabla H|^2 + \frac{1}{r}\nabla^m r \nabla_m |H|^2 \\
&+ 2H^\alpha \lambda_\alpha \left(\frac{2}{r^2}|\nabla r|^2 + H^\alpha \lambda_\alpha\right) + 4|H|^2\left(|H|^2 - \frac{1}{r^2}|\nabla r|^2\right) \\
=& 2|H|\Delta|H| - 2|H|^2\left(|H|^2 + |\tau|^2\right) + \frac{1}{r}\nabla^m r \nabla_m |H|^2 \\
&+ 2H^\alpha \lambda_\alpha \left(\frac{2}{r^2}|\nabla r|^2 + H^\alpha \lambda_\alpha\right) + 4|H|^2\left(|H|^2 - \frac{1}{r^2}|\nabla r|^2\right)
\end{aligned}$$

This gives

$$\begin{aligned}
\frac{\partial}{\partial t}|H| =& \Delta |H| + |H|\left(|H|^2 - |\tau|^2 - \frac{2}{r^2}|\nabla r|^2\right) \\
&+ \frac{1}{r}\nabla^m r \nabla_m |H| + \nu^\alpha \lambda_\alpha \left(\frac{2}{r^2}|\nabla r|^2 + H^\alpha \lambda_\alpha\right).
\end{aligned} \tag{4.6}$$

Proposition 4.9. *Let $t_n = \tilde{t_n} + \frac{1}{4\sigma \mathcal{M}_{\tilde{t_n}}}$ for some $\tilde{t_n} \in [0, t_n)$. Assume that $\rho \mathcal{M}_t \leqslant \mathcal{M}_{t_n}$ for all $t \leqslant t_n$. Then there exists a constant d depending only on ρ such that*

$$\left|\frac{\partial}{\partial t}|H|\right| \leqslant d {\mathcal{M}_{t_n}}^{\frac{3}{2}}. \tag{4.7}$$

for $t \in \left[t_n, t_n + \frac{\rho}{4\sigma \mathcal{M}_{t_n}}\right]$. As long as $\mathcal{N}_{t_n}$ is bounded from below, the inequality can be rewritten in terms of $\mathcal{N}_{t_n}$.

Proof. Assumption (4.2) and (4.6) lead to

$$\frac{\partial}{\partial t}|H| \leqslant |\nabla\nabla |H|| - |H||\tau|^2 + |H|^3 + d_2 |\nabla |H|| + d_1 |H| + d_0,$$

for some constants d_0, d_1, d_2. Corollary 4.8 imply that all summands can be bounded in terms of ${\mathcal{M}_{t_n}}^{\frac{3}{2}}$ independent of the time t_n, for example:

$$\left||\nabla\nabla |H|| - |H||\tau|^2\right| \leqslant \left|\nabla_j \nabla_i |H| - |H|^3 g_{ij} - |H|\tau_i \tau_j\right| + |H|^3 \leqslant \tilde{c}_3 {\mathcal{M}_{t_n}}^{\frac{3}{2}},$$

for some constant $\tilde{c}_3$. Note that the condition $\rho \mathcal{M}_0 \leqslant \mathcal{M}_{t_n}$ ensures that the lower order terms can be bounded in terms of ${\mathcal{M}_{t_n}}^{\frac{3}{2}}$ independently of t_n, too. □

Remark 4.10. It was shown in Corollary 2.5 that if t_n is large enough it is always possible to find an earlier time $\tilde{t_n}$ such that $t_n = \tilde{t_n} + \frac{1}{4\sigma \mathcal{M}_{\tilde{t_n}}}$.

These estimates will be used only for times t_n coming from an (essential) blow-up sequence, therefore, we can restrict ourselves to sufficiently large times t_n such that $\mathcal{N}_{t_n}$ is bounded from below by a firmly chosen constant, for example $\mathcal{N}_{\tilde{t_n}} \geqslant \frac{1}{r_0^2}$.

These bounds on the time derivative of the absolute curvature give now control on its total loss on small intervals of time and space. Therefore, a small neighbourhood of a point (p_n, t_n) where the curvature does not vanish is defined:

Definition 4.11. *For $d > 0$ and $(p_n, t_n) \in S^1 \times [0, T)$ define*

$$\mathbb{I}\left(p_n, t_n, d\right) := \left\{ (p, t) \in S^1 \times [t_n, T) \,; \mathrm{dist}_{t_n} \left\{p_n, p\right\} \leqslant \sqrt{\frac{d}{\mathcal{N}_{t_n}}}, (t - t_n) \leqslant \frac{d}{\mathcal{N}_{t_n}} \right\},$$

where the distance is defined in terms of the time dependent metric

$$\mathrm{dist}_{t_n} \left\{u_1, u_2\right\} = \left| \int_{u_1}^{u_2} d\mu\left(t_n\right) \right|.$$

As mentioned above, it is necessary to know that the curvature does not vanish 'near' a point where the curvature attains its maximum at some time t_n. The next lemma ensures that in small neighbourhoods defined as above, the curvature remains bounded from below. The first statement is that whenever the curvature does not vanish in a point $p_n \in S^1$ at some time t_n, it will not vanish in this point for small future times, i. e. the *temporal loss* is bounded. The second statement bounds the loss in *spatial* direction: if $|H(p_n, t)| \neq 0$, it remains positive on small space intervals on the right and left of p_n. These two statements are united in the third one in which the *total loss* is bounded. Altschuler proved this for curves under curve shortening flow in [4, Theorem 4.3]. To get the third statement out of the previous two, he used that the length of curves evolving under curve shortening flow decreases. For the perturbed flow, this is no longer satisfied, in fact, the length of curves may increase under the perturbed flow.

Lemma 4.12. *Let (p_n, t_n) be an essential blow-up sequence. Without loss of generality assume $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$. Then there exist constants $d_0, d_1, d_2 > 0$ depending only on ρ and r_0 such that the following holds:*

1. *The temporal loss of $|H(p_n, \cdot)|$ is bounded from below*

$$|H(p_n, t)| \geqslant \frac{|H(p_n, t_n)|}{\sqrt{2}} \text{ for } t \in \left[t_n, t_n + \frac{d_1}{\mathcal{N}_{t_n}}\right]. \tag{4.8}$$

2. *The spatial loss of $|H(\cdot, t)|$ is bounded from below*

$$|H(p, t)| \geqslant \frac{|H(p_n, t)|}{\sqrt{2}} \text{ for } \mathrm{dist}_t \left\{p, p_n\right\} \leqslant \sqrt{\frac{d_2}{\mathcal{N}_{t_n}}}, t \in \left[t_n, t_n + \frac{d_1}{\mathcal{N}_{t_n}}\right]. \tag{4.9}$$

3. The total loss of $|H(\cdot,\cdot)|$ *is bounded from below*

$$|H(p,t)| \geqslant \frac{|H(p_n,t_n)|}{2} \ \textit{for} \ (p,t) \in \mathbb{I}(p_n,t_n,d_0). \tag{4.10}$$

Proof. Let (p_n, t_n) be an essential blow-up sequence such that $t_n = \tilde{t_n} + \frac{1}{4\sigma \mathcal{M}_{\tilde{t_n}}}$ for some $\tilde{t_n}$ and $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$. Proposition 4.9 gives a bound on $\left|\frac{\partial}{\partial t}|H|\right|$ in terms of $\mathcal{N}_{t_n}$: $\left|\frac{\partial}{\partial t}|H|\right| \leqslant c_1 \mathcal{N}_{t_n}^{\frac{3}{2}}$ for $t \in \left[t_n, t_n + \frac{\rho}{8\sigma\mathcal{N}_{t_n}}\right] \subset \left[t_n, t_n + \frac{\rho}{4\sigma\mathcal{M}_{t_n}}\right]$, where c_1 depends only on ρ and r_0. Now, choose a constant $c_2 > 0$ independent of t_n, such that for $(t - t_n) \leqslant \min\left(\frac{c_2}{\mathcal{N}_{t_n}}, \frac{\rho}{8\sigma\mathcal{N}_{t_n}}\right)$

$$\begin{aligned}|H(p_n,t)| &\geqslant |H(p_n,t_n)| - (t-t_n)\, c_1 \mathcal{N}_{t_n}^{\frac{3}{2}} \geqslant |H(p_n,t_n)| - c_1 c_2 \mathcal{N}_{t_n}^{\frac{1}{2}} \\ &\geqslant |H(p_n,t_n)| - \frac{c_1 c_2}{\rho^{\frac{1}{2}}} |H(p_n,t_n)| \geqslant \frac{|H(p_n,t_n)|}{\sqrt{2}}.\end{aligned}$$

Defining $d_1 := \min\left\{\frac{\rho}{8\sigma}, c_2\right\}$ gives (4.8).

Corollary 4.8 implies the existence of a constant c_3 depending only on ρ such that $|\nabla|H||^2 \leqslant c_3 \mathcal{N}_{t_n}^2$. Since c is a space curve $|\nabla|H||^2 = \left|\frac{\partial|H|}{\partial x^1}\right|^2$, so the same argumentation as above yields

$$|H(p,t_n)| \geqslant \frac{|H(p_n,t_n)|}{\sqrt{2}} \ \text{for} \ \mathrm{dist}_t\{p,p_n\} \leqslant \frac{c_4}{\sqrt{\mathcal{N}_{t_n}}},$$

for some sufficiently small c_4. Defining $\sqrt{d_2} := \min\{d_1, c_4\}$ gives (4.9).

To get the third statement, note that in the definition of $\mathbb{I}(p_n,t_n,d_0)$ the metric is fixed at time t_n whereas in the second point of the theorem the metric depends on t. Therefore, it has to be ensured that the distance between two points do not change to fast on short time intervals. The time derivative of the length is given by

$$\frac{\partial}{\partial t} L(c) = -\int_c \left(|H|^2 + \langle H, \lambda\rangle\right) \mathrm{d}\mu \leqslant -\frac{1}{2}\int_c \left(|H|^2 - |\lambda|^2\right) \mathrm{d}\mu \leqslant \frac{1}{2r_0^2} L(c). \tag{4.11}$$

Since $|\lambda|^2 \leqslant \frac{1}{r_0^2}$ is bounded, the length can only increase exponentially in time, so the last assertion follows from the previous two. □

Remark 4.13. The length of a curve evolving under the perturbed flow must not decrease. In fact, (3.6b) implies

$$-\int_c \langle H, \lambda\rangle \,\mathrm{d}\mu = \int_c \frac{1}{r}\Delta r \,\mathrm{d}\mu = -\int_c \nabla^m\left(\frac{1}{r}\right)\nabla_m r \,\mathrm{d}\mu = \int_c \frac{1}{r^2}|\nabla r|^2 \,\mathrm{d}\mu \geqslant 0,$$

where equality holds if and only if $r(\cdot,t)$ is constant.

Hence, (4.11) becomes

$$\frac{\partial}{\partial t} L(c) = -\int_c \left(|H|^2 - \frac{1}{r^2} |\nabla r|^2 \right) \mathrm{d}\mu.$$

and the the length increases, if $|H|^2 < \frac{1}{r^2} |\nabla r|^2$. Since c is compact, $r(\cdot, t)$ is constant for all $0 \leqslant t < T$, if $r(\cdot, 0)$ is constant initially. Then, the length decreases during the flow.

Accordantly to Lemma 4.12, the total loss of the torsion τ can be bounded, too, at least as long as the curvature $|H| > \varepsilon > 0$ and (as always) $r \geqslant r_0$ are bounded away from zero. Note that here it is assumed that the curvature do not vanish on some interval of the form $\mathbb{I}(p_n, t_n, d)$. But this is not a restriction since the loss of curvature in such a interval is limited by Lemma 4.12. First, Proposition 4.5 gives bounds for the torsion and its derivatives:

Proposition 4.14. *Let $\{(p_n, t_n)\}$ be an essential blow-up sequence (such that $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$). Then there exist constants $d > 0$ and $\infty > \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$ depending only on ρ and r_0 such that the first derivatives of τ are bounded on the interval $\mathbb{I}(p_n, t_n, d)$:*

$$|\tau|^2 \leqslant \tilde{c}_1 \mathcal{M}_{t_n} \leqslant 2\tilde{c}_1 \mathcal{N}_{t_n}, \quad |\nabla\tau|^2 \leqslant \tilde{c}_2 {\mathcal{M}_{t_n}}^2 \leqslant 2\tilde{c}_2 \mathcal{N}_{t_n}^2, \quad |\nabla\nabla\tau|^2 \leqslant \tilde{c}_3 {\mathcal{M}_{t_n}}^3 \leqslant 2\tilde{c}_3 \mathcal{N}_{t_n}^3.$$

Proof. Corollary 4.8 and Lemma 4.12 imply

$$|\tau(p,t)|^2 \leqslant c_1 \frac{{\mathcal{M}_{t_n}}^2}{|H(p,t)|^2} \leqslant 2c_1 \frac{{\mathcal{M}_{t_n}}^2}{|H(p_n,t_n)|^2} \leqslant \frac{2c_1}{\rho} \mathcal{M}_{t_n},$$

for $(p,t) \in \mathbb{I}(p_n, t_n, d)$. Proceeding like this, we get the remaining inequalities:

$$\begin{aligned} |\nabla\tau| \leqslant & \frac{1}{|H|} \left(\left| 2\nabla_j |H| \tau_i + |H| \nabla_j \tau_i \right| + |2\nabla_j |H|| \, |\tau_i| \right) \leqslant c_2 \frac{{\mathcal{M}_{t_n}}^{\frac{3}{2}}}{|H|} \leqslant \frac{2c_2}{\sqrt{\rho}} \mathcal{M}_{t_n}, \\ |\nabla\nabla\tau| \leqslant & \frac{1}{|H|} \left| 3\nabla_m \nabla_j |H| \tau_i + 3\nabla_j |H| \nabla_m \tau_i + |H| \nabla_j \nabla_m \tau_i - |H|^3 g_{ij} \tau_m - |H| \tau_m \tau_i \tau_j \right| \\ & + \frac{1}{|H|} |\tau| \left| 3\nabla_i \nabla_j |H| - |H|^3 g_{ij} - |H| \tau_i \tau_j \right| + 3 \frac{1}{|H|} |\nabla_j |H| \nabla_m \tau_i| \leqslant \frac{2c_3}{\sqrt{\rho}} {\mathcal{M}_{t_n}}^{\frac{3}{2}}. \end{aligned}$$

□

The evolution equations of the torsion and its norm (cf. Lemma B.4 and (B.17)) imply:

$$\begin{aligned} \left| \frac{\partial}{\partial t} |\tau|^2 \right| \leqslant & |\tau| \, |\nabla\nabla\tau| + |\tau| \, |\nabla\tau| \left(\frac{c_1}{|H|} |\nabla |H|| + c_2 \right) \\ & + |\tau|^2 \left(\frac{c_3}{|H|} |\nabla\nabla H| + \frac{c_4}{|H|^2} |\nabla |H||^2 + c_5 |H|^2 + \frac{c_6}{|H|} + c_7 \right) \\ & + |\tau| \left(\frac{c_8}{|H|^2} |\nabla |H|| + c_9 |H| + \frac{c_{10}}{|H|} + \mathsf{d}_{11} \right). \end{aligned} \tag{4.12}$$

As in the proof of Lemma 4.12, Proposition 4.14 and (4.12) lead to the following lemma concerning the loss of the torsion (cp. Theorem 4.13 stated in [4]). For completeness, the first step is outlined here.

Lemma 4.15. *Let (p_n, t_n) be an essential blow-up sequence (such that $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$). Assume that there exists a constant $\mu > 0$ such that $|\tau(p_n, t_n)|^2 \geqslant \mu |H(p_n, t_n)|^2$ holds for all n. Then there exist constants $d_3, d_4, d_5 > 0$ depending only on μ, ρ and r_0 such that the following hold:*

1. *The temporal loss of $|\tau(p_n, \cdot)|$ is bounded from below*
$$|\tau(p_n, t)|^2 \geqslant \frac{|\tau(p_n, t_n)|^2}{\sqrt{2}} \text{ for } t \in \left[t_n, t_n + \frac{d_4}{\mathcal{N}_{t_n}}\right]. \tag{4.13}$$
2. *The spatial loss of $|\tau(\cdot, t)|$ is bounded from below*
$$|\tau(p, t)|^2 \geqslant \frac{|\tau(p_n, t)|^2}{\sqrt{2}} \text{ for } \operatorname{dist}_t\{p, p_n\} \leqslant \sqrt{\frac{d_5}{\mathcal{N}_{t_n}}}, t \in \left[t_n, t_n + \frac{d_4}{\mathcal{N}_{t_n}}\right]. \tag{4.14}$$
3. *The total loss of $|\tau(\cdot, \cdot)|$ is bounded from below*
$$|\tau(p, t)|^2 \geqslant \frac{|\tau(p_n, t_n)|^2}{2} \text{ for } (p, t) \in \mathbb{I}(p_n, t_n, d_3) \tag{4.15}$$

Proof. Let (p_n, t_n) be an essential blow-up sequence and assume $(p, t) \in \mathbb{I}(p_n, t_n, d)$ for a sufficiently small d depending only on ρ and r_0 such that Lemma 4.12 is valid on $\mathbb{I}(p_n, t_n, d)$. By Corollary 4.8, Proposition 4.14 and (4.12), there exists a constant $c_1 > 0$ depending only on ρ and r_0 such that

$$\left|\frac{\partial}{\partial t}|\tau|^2\right| \leqslant c_1 \mathcal{N}_{t_n}^2 \text{ for } (p, t) \in \mathbb{I}(p_n, t_n, d),$$

because all summands on the right hand side of (4.12) can be bounded: for example, $\frac{1}{|H|}|\nabla\nabla H| \leqslant \frac{c}{|H|(p_n, t_n)}\mathcal{M}_{t_n}{}^{\frac{3}{2}} \leqslant \frac{c}{\sqrt{\rho}}\mathcal{M}_{t_n}$. Since (p_n, t_n) is essential, terms of lower order can be bounded, too. Choosing a proper constant $c_2 > 0$, we get

$$\begin{aligned}
|\tau(p_n, t)|^2 &\geqslant |\tau(p_n, t_n)|^2 - (t - t_n) c_1 \mathcal{N}_{t_n}^2 \geqslant |\tau(p_n, t_n)|^2 - c_1 c_2 \mathcal{N}_{t_n} \\
&\geqslant |\tau(p_n, t_n)|^2 - \frac{c_1 c_2}{\rho}|H(p_n, t_n)|^2 \geqslant |\tau(p_n, t_n)|^2 - \frac{c_1 c_2}{\rho\mu}|\tau(p_n, t_n)|^2 \\
&\geqslant \frac{|\tau(p_n, t_n)|^2}{\sqrt{2}} \text{ for } 0 \leqslant t - t_n \leqslant \frac{c_2}{\mathcal{N}_{t_n}}.
\end{aligned}$$

Note that c_2 only depends on ρ and μ. Setting $d_4 := \min\left\{c_2, \frac{\rho}{2c\mathcal{N}_{t_n}}\right\}$ gives the first statement.

Since c is a curve $\left|\nabla |\tau|^2\right|^2 = |\tau|^2 |\nabla\tau|^2$ and this is bounded by Corollary 4.8. Hence the second and third statement follow in the same way as in the proof to Lemma 4.12. □

4.2. The total curvature

The argumentation continues to follow [4]. There, Altschuler proved in Theorem 5.1 a monotonicity for the total curvature for curves under mean curvature flow. For the perturbed flow, such a monotonicity is no longer valid, but as long as λ remains bounded the total curvature do not increase to fast. This will be shown in Lemma 4.17. In the proof, the following proposition will be needed which is a well known result from the theory of ODEs.

Proposition 4.16. *If $f : \mathbb{R} \to \mathbb{R}$ satisfies $\frac{\partial}{\partial t} f \leqslant Af + B \exp(Ct)$ with constants A, B, C, there exists a constant M such that*

$$\begin{cases} f \leqslant M \exp(At) + \frac{B}{C-A} \exp(Ct) & \text{if } A \neq C \\ f \leqslant M \exp(At) + Bt \exp(At) & \text{if } A = C \end{cases}$$

Proof. Differentiation of $\exp(-At) f$ gives

$$\frac{\partial}{\partial t} (f \exp(-At)) \leqslant \exp(-At)(Af + B\exp(Ct)) - A\exp(-At) f = B \exp((C-A)t)$$

and the proposition follows by integration. □

Lemma 4.17. *Let c be a solution to the perturbed mean curvature flow* (4.1) *satisfying assumption* (4.2). *The total mean curvature $\int_c |H| \, d\mu$ is integrable over $[0,T)$ and*

$$\frac{\partial}{\partial t} \int_c |H| \, d\mu \leqslant -\int_c |H| \, |\tau|^2 \, d\mu + \frac{1}{r_0^2} \int_c |H| \, d\mu + \frac{2}{r_0^3} L(c). \tag{4.16}$$

Proof. First note, that $\int_c |H| \, \mathrm{d}\mu$ is differentiable on $[0,T)$, even though there may occur (isolated) points in $S^1 \times [0,T)$ where $|H|$ is not differentiable. To avoid technical difficulties, define $K_\epsilon := \sqrt{|H|^2 + \epsilon}$ for an $\epsilon > 0$. The convergence $K_\epsilon \to |H|$ is uniformly for $\epsilon \to 0$ and, because the length $L(c)$ grows at most exponentially (cf. (4.11)), the convergence $\int_c K_\epsilon \, \mathrm{d}\mu \to \int_c |H| \, \mathrm{d}\mu$ is also uniformly, whereas $\frac{\partial}{\partial t} \int_c K_\epsilon \, \mathrm{d}\mu \to \frac{\partial}{\partial t} \int_c |H| \, \mathrm{d}\mu$ is only a pointwise convergence.

Since

$$\Delta K_\epsilon = g^{ij} \nabla_i \left(\frac{1}{2K_\epsilon} \cdot \nabla_j |H|^2 \right) = \frac{1}{2K_\epsilon} \Delta |H|^2 - \frac{g^{ij} \nabla_i |H|^2 \nabla_j |H|^2}{4K_\epsilon^3},$$

the evolution equation (3.12) of $|H|^2$ implies

$$\begin{aligned} \frac{\partial}{\partial t} K_\epsilon =& \frac{1}{2K_\epsilon} \frac{\partial}{\partial t} |H|^2 \\ =& \Delta K_\epsilon + \frac{\left| \nabla |H|^2 \right|^2}{4K_\epsilon^3} + \frac{1}{2K_\epsilon} \left(-2 |\nabla H|^2 + \frac{1}{r} \nabla^m r \nabla_m |H|^2 \right) \\ &+ \frac{1}{2K_\epsilon} \left(2H^\alpha \lambda_\alpha \left(\frac{2}{r^2} |\nabla r|^2 + H^\alpha \lambda_\alpha \right) + 4 |H|^2 \left(|H|^2 - \frac{1}{r^2} |\nabla r|^2 \right) \right). \end{aligned}$$

This gives

$$\begin{aligned}\frac{\partial}{\partial t}\int_c K_\epsilon \mathrm{d}\mu =& \int_c \Delta K_\epsilon + \frac{\left|\nabla |H|^2\right|^2}{4K_\epsilon^3} - \frac{1}{K_\epsilon}|\nabla H|^2 + \frac{1}{r}\nabla^m r \nabla_m K_\epsilon \,\mathrm{d}\mu \\ &+ \int_c \frac{1}{2K_\epsilon}\left(2H^\alpha\lambda_\alpha\left(\frac{2}{r^2}|\nabla r|^2 + H^\alpha\lambda_\alpha\right)\right)\mathrm{d}\mu \\ &+ \int_c \frac{2|H|^2}{K_\epsilon}\left(\left(|H|^2 - \frac{1}{r^2}|\nabla r|^2\right)\right)\mathrm{d}\mu - \int_c K_\epsilon\left(|H|^2 + H^\alpha\lambda_\alpha\right)\mathrm{d}\mu.\end{aligned}$$

Integration by parts yields

$$\begin{aligned}\frac{\partial}{\partial t}\int_c K_\epsilon \mathrm{d}\mu =& \int_c \frac{\left|\nabla |H|^2\right|^2}{4K_\epsilon^3} - \frac{1}{K_\epsilon}|\nabla H|^2 + K_\epsilon\left(\frac{1}{r^2}|\nabla r|^2 + H^\alpha\lambda_\alpha\right)\mathrm{d}\mu \\ &+ \int_c \frac{1}{2K_\epsilon}\left(2H^\alpha\lambda_\alpha\left(\frac{2}{r^2}|\nabla r|^2 + H^\alpha\lambda_\alpha\right)\right)\mathrm{d}\mu \\ &+ \int_c \frac{2|H|^2}{K_\epsilon}\left(\left(|H|^2 - \frac{1}{r^2}|\nabla r|^2\right)\right)\mathrm{d}\mu - \int_c K_\epsilon\left(|H|^2 + H^\alpha\lambda_\alpha\right)\mathrm{d}\mu,\end{aligned}$$

where we used $\Delta r = -r\langle H, \lambda\rangle$. Restriction to areas where the curvature vector $H \neq 0$ does not change the value of the integral and the Frenet equalities (3.13) lead to

$$\begin{aligned}\frac{\partial}{\partial t}\int_c K_\epsilon \mathrm{d}\mu =& \int_{|H|\neq 0} \frac{|H|^2}{K_\epsilon^3}|\nabla |H||^2 - \frac{1}{K_\epsilon}\left(|\nabla |H||^2 - |H|^2\left(|H|^2 - |\tau|^2\right)\right) - K_\epsilon |H|^2 \mathrm{d}\mu \\ &+ \int_{|H|\neq 0} K_\epsilon \frac{1}{r^2}|\nabla r|^2 - \frac{2|H|^2}{K_\epsilon}\frac{1}{r^2}|\nabla r|^2 + \frac{1}{K_\epsilon}H^\alpha\lambda_\alpha\left(\frac{2}{r^2}|\nabla r|^2 + H^\alpha\lambda_\alpha\right)\mathrm{d}\mu.\end{aligned}$$

Furthermore, $|H|^2 \leqslant K_\epsilon^2$ and the Cauchy-Schwarz inequality imply

$$\frac{\partial}{\partial t}\int_c K_\epsilon \mathrm{d}\mu \leqslant \int_c -\frac{|H|^2}{K_\epsilon}\left(|\tau|^2 + \frac{2}{r^2}|\nabla r|^2 - |\lambda|^2\right) + K_\epsilon\frac{1}{r^2}|\nabla r|^2 + |\lambda|\frac{2}{r^2}|\nabla r|^2 \,\mathrm{d}\mu.$$

Since $\frac{|H|^2}{K_\epsilon} \to |H|$ uniformly, inequality (4.16) follows from the pointwise convergence of the time derivatives for $\epsilon \to 0$:

$$\frac{\partial}{\partial t}\int |H|\,\mathrm{d}\mu \leqslant -\int |H|\,|\tau|^2\,\mathrm{d}\mu + \int\left(|\lambda|\left(\frac{2}{r^2}|\nabla r|^2 + |H|\,|\lambda|\right) + |H|\frac{1}{r^2}|\nabla r|^2\right)\mathrm{d}\mu. \tag{4.17}$$

Thus, inequality (4.16) follows from $|\lambda| \leqslant \frac{1}{r_0}$ and $|\nabla r|^2 = 1 - r^2|\lambda|^2 \leqslant 1$.

Again, the at most exponential growth of the length implies that $\int_c |H|\,\mathrm{d}\mu$ is integrable over the finite time interval $[0, T)$, moreover, it grows at most exponentially, too, and the limit $\lim_{t\to T}\int_c |H|\ \mathrm{d}\mu$ exists: inserting $L(c_t) \leqslant L(c_0)\exp\left(\frac{1}{2r_0^2}t\right)$ (cf. (4.11)) in (4.16) gives

$$\frac{\partial}{\partial t}\int_c |H|\,\mathrm{d}\mu \leqslant \frac{1}{r_0^2}\int_c |H|\,\mathrm{d}\mu + \frac{2}{r_0^3}L(c_0)\exp\left(\frac{1}{2r_0^2}t\right).$$

By Proposition 4.16, there exists a constant M such that

$$\int_c |H|\ \mathrm{d}\mu \leqslant M\exp\left(\frac{1}{r_0^2}t\right) - \frac{L(c_0)}{r_0}\exp\left(\frac{1}{2r_0^2}t\right).$$

Hence, $\int_c |H|\ \mathrm{d}\mu$ and $\frac{\partial}{\partial t}\int_c |H|\ \mathrm{d}\mu$ are integrable on the finite time interval $[0,T)$. □

The next corollary follows immediately from Lemma 4.17

Corollary 4.18.

$$\int_0^T \int_C |H|\,|\tau|^2\, d\mu\, dt < \infty \tag{4.18}$$

Proof. At the end of the proof of Lemma 4.17, it was shown that $\int |H|\,\mathrm{d}\mu$, $\frac{\partial}{\partial t}\int |H|\,\mathrm{d}\mu$ and $L(c)$ are integrable over $[0,T]$. Hence, we have

$$\begin{aligned}&\int_0^T \int_C |H|\,|\tau|^2\,\mathrm{d}\mu \\ &\quad\leqslant \frac{1}{r_0^2}\int_0^T\int_C |H|\,\mathrm{d}\mu\mathrm{d}t + \frac{2}{r_0^3}\int_0^T L(c_t)\ \mathrm{d}t + \int_C |H|\,\mathrm{d}\mu\,(0) - \int_C |H|\,\mathrm{d}\mu\,(T) < \infty.\end{aligned}$$

□

Corollary 4.18 will be used to control the torsion whenever the curvature vanishes, therefore, it is written in the form

Corollary 4.19. *For all $\epsilon > 0$, there exits a $\theta > 0$ such that*

$$\int_{T-\theta}^T \int_C |H|\,|\tau|^2\, d\mu < \epsilon.$$

4.3. Forming singularities become planar

As in [4], the dilation-invariant estimates and the integral estimates will now be used to study the singular behaviour of solutions to the perturbed mean curvature flow (4.1). First, it is proved that the solutions become planar in the sense of Definition 3.5.

Theorem 4.20. *Assume that condition* (4.2) *is satisfied:* $r \geqslant r_0 > 0$ *on* $[0, T)$. *Without loss of generality let* $(p_n, t_n) \in S^1 \times [0, T)$ *be an essential blow-up sequence such that* $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$ *(see Definition 4.3). Then*

$$\lim_{n \to \infty} \frac{|\tau(p_n, t_n)|}{|H(p_n, t_n)|} = 0. \tag{4.19}$$

Proof. Suppose there is a subsequence (p_{n_k}, t_{n_k}) and a lower bound $\mu > 0$ such that

$$|\tau(p_{n_k}, t_{n_k})|^2 \geqslant \mu |H(p_{n_k}, t_{n_k})|^2 \text{ for all } k \in \mathbb{N}.$$

For simplicity, the subscript k will be omitted, and (p_n, t_n) will be written instead of (p_{n_k}, t_{n_k}).

First note, that Remark 4.10, Lemma 4.12 and Lemma 4.15 prevent the curvature and the torsion of decreasing too quickly in time and space: there exists a sufficient small constant $d > 0$ depending only on ρ and r_0 such that for $(p, t) \in \mathbb{I}(p_n, t_n, d)$, see Definition 4.11,

$$|H(p,t)| \geqslant \frac{1}{2} |H(p_n, t_n)| \tag{4.20a}$$

$$|\tau(p,t)|^2 \geqslant \frac{1}{2} |\tau(p_n, t_n)|^2 . \tag{4.20b}$$

Particularly, this implies $\mathcal{N}_t \geqslant |H(p,t)| \geqslant \frac{\rho}{2}\mathcal{N}_s$, for all $s \leqslant t_n$ and $(p,t) \in \mathbb{I}(p_n, t_n, d)$.

Secondly, distances along the curves c do not decrease too fast, in fact the loss is at most exponential: fix a time t and two points q_1, q_2 such that $(q_1, t), (q_2, t) \in \mathbb{I}(p_n, t_n, d)$. The volume form of c evolves by (cf. (3.10))

$$\frac{\partial}{\partial t} \mathrm{d}\mu = -\Big(|H|^2 + \langle H, \lambda \rangle\Big) \mathrm{d}\mu.$$

Since (p_n, t_n) is an essential blow-up sequence, $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$ implies $|H(p_n, t_n)|^2 \geqslant \frac{\rho}{r_0^2}$. Hence, (4.20a) yields

$$|H(q_i,t)| \, |\lambda(q_i,t)| \leqslant \frac{1}{r_0} |H(q_i,t)| \leqslant |H(q_i,t)| \frac{|H(p_n,t_n)|}{\sqrt{\rho}} \leqslant \frac{2}{\sqrt{\rho}} |H(q_i,t)|^2 \leqslant \frac{2}{\sqrt{\rho}} \mathcal{N}_t.$$

The previous estimates together with the Cauchy-Schwarz inequality lead to

$$\begin{aligned} \frac{\partial}{\partial t} (\mathrm{dist}_t \{q_1, q_2\}) &= -\int_{(q_1,q_2)} \Big(|H|^2 + \langle H, \lambda \rangle\Big) \mathrm{d}\mu \geqslant -d_1 \mathcal{N}_t \int_{(q_1,q_2)} \mathrm{d}\mu \\ &\geqslant -d_2 \mathcal{N}_{t_n} \mathrm{dist}_t \{q_1, q_2\}, \end{aligned}$$

with some constants d_i depending only on ρ. Integration gives

$$\mathrm{dist}_t \{q_1, q_2\} \geqslant \mathrm{dist}_{t_n} \{q_1, q_2\} \, e^{-d_2(\rho)\mathcal{N}_{t_n}(t - t_n)} \geqslant \mathrm{dist}_{t_n} \{q_1, q_2\} \, e^{-d_3},$$

since $(t-t_n) \leqslant \frac{d}{\mathcal{N}_{t_n}}$ by the choice of $\mathbb{I}(p_n, t_n, d)$ (cp. Definition 4.11). Therefore, the change of distance is not too big on sufficiently small time intervals.

Hence, there exists a constant $d_4 > 0$ depending only on μ, ρ, d, and r_0 such that the integral

$$\begin{aligned}\int_{t_n}^{t_n+\frac{1}{8\sigma\mathcal{N}_{t_n}}} \int_c |\tau|^2 |H| \,\mathrm{d}\mu\mathrm{dt} &\geqslant \iint_{\mathbb{I}(p_n,t_n,d)} |\tau|^2 |H| \,\mathrm{d}\mu\mathrm{dt} \geqslant \mu \left(\frac{\rho}{2}\mathcal{N}_{t_n}\right)^{\frac{3}{2}} \iint_{\mathbb{I}(p_n,t_n,d)} \mathrm{d}\mu\mathrm{dt} \\ &\geqslant \mu \left(\frac{\rho}{2}\mathcal{N}_{t_n}\right)^{\frac{3}{2}} e^{-d_3} \sqrt{\frac{d_0}{\mathcal{N}_{t_n}}} \frac{d}{\mathcal{N}_{t_n}} \geqslant \mu \left(\frac{d\rho}{2}\right)^{\frac{3}{2}} e^{-d_3} =: d_4 \end{aligned} \tag{4.21}$$

is uniformly bounded away from zero. The third inequality is valid by the definition of $\mathbb{I}(p_n, t_n, d_0)$.

But this contradicts Corollary 4.19: for every $d_4 > \epsilon > 0$, there exists a θ such that

$$\int_{T-\theta}^{T} \int_C |\tau|^2 |H| \,\mathrm{d}\mu\,\mathrm{dt} \leqslant \int_{t_n}^{t_n+\frac{1}{8\sigma\mathcal{N}_{t_n}}} \int_C |\tau|^2 |H| \,\mathrm{d}\mu\,\mathrm{dt} \leqslant \int_{T-\theta}^{T} \int_C |\tau|^2 |H| \,\mathrm{d}\mu\,\mathrm{dt} \leqslant \epsilon, \tag{4.22}$$

for n sufficiently large. □

4.3.1. The limit of rescaled solutions

The dilation-invariant estimates for curves evolving under the perturbed flow (see Proposition 4.5 and Corollary 4.6) also imply the convergence to a limit solution after appropriate rescaling. The limit solution is a family of planar curves moving by the ordinary mean curvature flow. Since the generated surfaces M_t evolve under mean curvature flow, translating the surfaces does not change the flow. Contrary, this is not true for the generating curves: the perturbed flow is not invariant under translations in direction e_1.

The rescaling is focused on the surfaces and divided into three steps: first, translate a point $F(p_n, 0, t_n)$ on the surface and therefore, the corresponding point on the generating curve $c(p_n, t_n)$ into the origin (see Figure 4.4. and Figure 4.5.), secondly, rescale the surface such that the norm of the mean curvature vector of the rescaled surface in the origin is bounded.

Translating and rescaling of M in the e_1-direction lead to a translation of the axes of symmetry. Fixing the euclidean coordinates of $\mathbb{R}^3$ implicates that the profile curve defined as $c_n := \mathbb{R}^3 \cap M_n$ must not lie in $\mathbb{R}^3_{>0}$ any longer. But the flow of each M_n is still determined by c_n, each evolving under a new flow. So, the third step is to determine the new flow for each c_n.

Definition 4.21. *The* rescaled solutions F_n *of* F *and* c_n *of* c *along an essential blow-up sequence* $(p_n, t_n) \in S^1 \times [0, T)$ *for* c *(resp. the corresponding essential blow-up sequece* $(p_n, 0, t_n) \in S^1 \times S^1 \times [0, T)$ *for* F*) are defined as follows:*

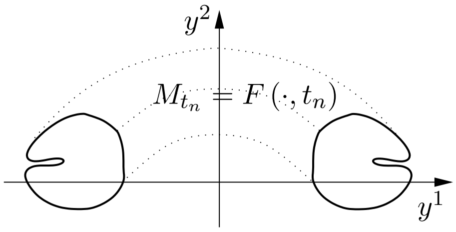

Figure 4.4.: M_{t_n} symmetric with respect to the y^2-axis

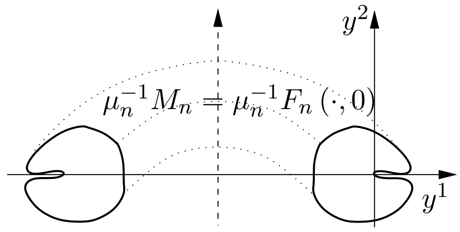

Figure 4.5.: Translating M_{t_n} changes the symmetry-axis

For $\bar{t} := \mu_n^2 (t - t_n)$ *and* $I_n := \left[-\mu_n^2 t_n^2, \mu_n^2 (T - t_n)\right)$, *set*

$$F_n : S^1 \times S^1 \times I_n \to \mathbb{R}^4, \quad F_n(x, \phi, \bar{t}) := \mu_n \left(F(x, \phi, t) - F(p_n, 0, t_n)\right), \tag{4.23a}$$

$$c_n : S^1 \times I_n \to \mathbb{R}^3, \quad c_n(x, \bar{t}) := \mu_n \left(c(x, t) - c(p_n, t_n)\right), \tag{4.23b}$$

where μ_n *is chosen sufficiently large such that* $|\mathcal{H}_n(p_n, 0, 0)|^2 = 1$, *i. e.*

$$\mu_n := |\mathcal{H}(p_n, t_n)| = |H(p_n, t_n) + \lambda(p_n, t_n)|,$$

where $\mathcal{H}_n$ *is the mean curvature vector of* $M_n(\bar{t}) := F_n\left(S^1 \times S^1 \times \{\bar{t}\}\right)$.

Since mean curvature flow is invariant under translations and time is scaled as space squared, F_n still evolves under the mean curvature flow:

$$\frac{\partial}{\partial \bar{t}} F_n = \mathcal{H}_n.$$

Furthermore, each M_n is still rotationally symmetric, and so its behaviour is fully determined by $c_n = M_n \cap \mathbb{R}^3$. But in general, the plane of rotation for M_n and M_m differs, therefore each c_n evolves under a different flow:

$$\frac{\partial}{\partial \bar{t}} c_n = H_n - \frac{1}{r_n + \mu_n r_{p_n}} e_1^\perp =: H_n + \lambda_n, \tag{4.24}$$

where H_n is the mean curvature vector of c_n, $r_n := \langle c_n, e_1 \rangle$ and $r_{p_n} := \langle c(p_n, t_n), e_1 \rangle$.

Remark 4.22. Note that the coordinates of the ambient euclidean space are fixed along the rescaling-process, so in general, $c_n \subset \mathbb{R}^3$ is no longer a subset of $\mathbb{R}^3_{>0}$, but $c \subset \left\{y \in \mathbb{R}^3; y^1 > -\mu_n r_{p_n}\right\}$, since $r_{p_n} \geqslant r_0 > 0$. This also implies $\lambda_n \neq -\frac{1}{r_n} e_1^\perp$.

Furthermore, the boundedness of $r \geqslant r_0$ away from zero implies that the radius of the surfaces M_n go to ∞ and planes of rotation to $-\infty$ for $n \to \infty$.

As mentioned above, the plane of rotation goes to $-\infty$, and also the length of the curves c_n might increase to ∞. Therefore, it is more convenient to think of F_n and c_n as periodic functions $F : \mathbb{R} \times \mathbb{R} \times [0, T) \to \mathbb{R}^4$ and $c : \mathbb{R} \times [0, T) \to \mathbb{R}^3_{>0}$ and choose parametrizations $u = u(x)$ and $\psi = \psi(\phi)$ such that each F_n and c_n are passed at time $\bar{t}$ with a fixed velocity:

$$F_n : \mathbb{R} \times \mathbb{R} \times I_n \to \mathbb{R}^4, \qquad F_n(u, \psi, \bar{t}) := \mu_n \left(F(x, \phi, t) - F(p_n, 0, t_n) \right) \tag{4.25a}$$

$$c_n : \mathbb{R} \times I_n \to \mathbb{R}^3, \qquad c_n(u, \bar{t}) := \mu_n \left(c(x, t) - c(p_n, t_n) \right) \tag{4.25b}$$

$$\frac{\partial}{\partial u} F_n = \frac{1}{\mu_n} \frac{\partial}{\partial x} F_n = \frac{\partial}{\partial x} F, \qquad \frac{\partial}{\partial \psi} F_n = \frac{1}{\mu_n} \frac{\partial}{\partial \phi} F_n = \frac{\partial}{\partial \phi} F, \quad \frac{\partial}{\partial u} c_n = \frac{1}{\mu_n} \frac{\partial}{\partial x} c_n = \frac{\partial}{\partial x} c, \tag{4.25c}$$

in fact, $\psi = \mu_n \phi$ and $u = \mu_n x$, i. e. the parameters is stretched. Furthermore, these are regular parametrizations, since $\mu_n \neq 0$ for all n.

Singularities occurring under mean curvature flow were characterized by Huisken in [38] and Angenent in [9].

Definition 4.23. *A forming singularity is of* type-I, *if* $\lim_{t \to T} \mathcal{M}_t \cdot (T - t)$ *is bounded. Otherwise it is of* type-II.

Under assumption (4.2) that the singularity occurs away from the plane of rotation, replacing $\mathcal{M}_t$ by $\mathcal{N}_t$ gives equivalent conditions.

Now, it is shown that the pertubation term λ do not influence the convergence of the rescaled solution as long as (4.2) is satisfied (cp. [4, Theorem 7.3]).

Theorem 4.24. *Let $(p_n, t_n) \in S^1 \times [0, T)$ be an essential blow-up sequence. Then for every compact set $K \subset \mathbb{R} \times (-\infty, 0]$ there exists a subsequence of (p_n, t_n) along which the rescaled solutions c_n considered as periodic functions on $\mathbb{R}$ converge in $C^\infty(K)$ to a smooth, nontrivial limit c_∞. The limit c_∞ is a solution to the ordinary mean curvature flow and it exists at least on the time interval $(-\infty, 0]$.*

Proof. Without loss of generality, assume that (p_n, t_n) is an essential blow-up sequence for c as well as for F with $\rho \mathcal{N}_s \leqslant H(p_n, t_n)$, $\rho \mathcal{M}_s \leqslant \mathcal{A}(p_n, t_n)$, for $s \leqslant t_n$, and $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$.

From $\mathcal{M}_{t_n} \to \infty$, it follows that $\lim_{n \to \infty} \left(-\mu_n^2 t_n \right) = -\infty$, whereas the limit of the upper bound of the rescaled time interval $\lim_{n \to \infty} \omega_n = \lim_{n \to \infty} \mu_n^2 (T - t_n)$ depends on the type of the occurring singularity. If the singularity is of type-I, $\lim_{n \to \infty} \mu_n^2 (T - t_n)$ is finite, whereas for a type-II singularity the limit depends on the choice of the blow-up sequence (p_n, t_n), which will be chosen in that way such that $\lim_{n \to \infty} \mu_n^2 (T - t_n) = \infty$.

As mentioned above, the length of the rescaled curves might be increasing to infinity, and hence, it is more convenient to think of the curves as periodic curves

$$c_n : \mathbb{R} \times \left[-\mu_n^2 t_n^2, \mu_n^2 (T - t_n) \right) \to \mathbb{R}^3$$

with $c_n(0, \cdot) = c_n(p_n, \cdot)$. Since the proof consists of an Arzelà-Ascoli argument applied to the tangent vectors T_n of the rescaled curves $c_n : \mathbb{R} \times \left[-\mu_n^2 t_n^2, 0 \right] \to \mathbb{R}^3$, it is necessary to parametrize c_n by u as in (4.25), hence the tangent vectors $T_n(u, \bar{t})$ have the same length for all n:

$$T_n := \frac{\partial}{\partial u} c_n := \frac{1}{\mu_n} \frac{\partial}{\partial x} c_n \left(= \frac{\partial}{\partial x} c \right), \quad |T_n(u, \bar{t})| = v(\bar{t}) := \left| \frac{\partial c}{\partial x} \right|.$$

First, all higher derivatives $\left|\nabla^N H_n\left(u, \bar{t}\right)\right|^2$ are bounded for all $-\mu_n^2 t_n < \bar{t} \leqslant 0$. Fix $\bar{t} \in \left(-\mu_n^2 t_n, 0\right]$. For the corresponding $0 < t \leqslant t_n$, there exists some $s \in (0, t_n)$ such that $t \in \left(s, s + \frac{1}{2\sigma \mathcal{M}_s}\right]$. Proposition 4.5 shows that the mean curvature vector and its higher derivatives are bounded:

$$\begin{aligned}\rho^{N+1}\left|\nabla^N H_n\left(\bar{t}\right)\right|^2 &= \frac{\rho^{N+1}}{\mu_n^{2N+2}}\left|\nabla^N H\left(t\right)\right|^2 \leqslant \frac{\rho^{N+1}}{\mu_n^{2N+2}}\left(\left|\bar{\nabla}^N \mathcal{A}\left(t\right)\right|^2\right)\\ &\leqslant \tilde{c_N}\frac{\rho^{N+1}}{\mu_n^{2N+2}}{\mathcal{M}_s}^{N+1} \leqslant 2\tilde{c_N}\frac{\left|H\left(p_n, t_n\right)\right|^{2N+2}}{\mu_n^{2N+2}} \leqslant c_N,\end{aligned} \tag{4.26}$$

for $N \geqslant 0$ and $\bar{t} \leqslant 0$. Note that the constants $\tilde{c_N}$ do not depend on s, and so all c_N are independent of $\bar{t}$ and n.

Secondly, use $r \geqslant r_0$ on $[0, T)$ and $\mu_n = \left|\mathcal{H}\left(p_n, t_n\right)\right| \geqslant \sqrt{\rho \mathcal{N}_{t_n}} \geqslant \frac{\sqrt{\rho}}{r_0}$ to bound the vector coming from the rotation $-\frac{1}{r_n + \mu_n r_{p_n}} e_1^{\perp}$ independent of n: $\frac{1}{r_n + \mu_n r_{p_n}} = \frac{1}{(\mu_n r)^2} \leqslant \frac{\rho}{r_0^4}$. Hence, the dilation-invariant estimates above give bounds on all higher (covariant) time derivatives of the tangent vector: $\left|\frac{\partial^M}{\partial \bar{t}^M} T_n\right|^2$, with $T_n = \frac{\partial}{\partial u} c_n$. Furthermore, all higher mixed derivatives consisting of N covariant derivatives in time and M in space are bounded independent of n:

$$\left|\nabla^{N+M} T_n\left(\frac{\partial}{\partial \bar{t}}, \dots \frac{\partial}{\partial \bar{t}}, \frac{\partial}{\partial u} \dots \frac{\partial}{\partial u}\right)\right|^2 \leqslant d_{N+M}.$$

Note that the parametrization u of c_n is chosen such that the length of a subarc with endpoints $c_n(u)$ and $c_n(v)$ is fixed for all n. Furthermore, the rescaling ensures that the distance a point $c_n\left(u, \bar{t}\right)$ can move during the flow on c_n is bounded. Since c is a curve, these estimates imply convergence of T_n along a subsequence on each compact subset $K \subset \mathbb{R} \times (-\infty, 0]$ via the Arzelà-Ascoli theorem: there exists a subsequence of (p_n, t_n) on which the tangent vectors $T_n\left(u, \bar{t}\right)$ converge uniformly on K to a smooth limit $T_\infty\left(u, \bar{t}\right)$. Integration gives a smooth limit c_∞.

Since $\left|\lambda_n\right|^2 \to 0$ uniformly in n, c_∞ is a solution to the mean curvature flow. Furthermore, the process of rescaling prevents the solution to be trivial:

$$\left|H_\infty\left(0, 0\right)\right|^2 = \lim_{n \to \infty}\left|H_n\left(0, 0\right)\right|^2 = \lim_{n \to \infty}\frac{\left|H\left(p_n, t_n\right)\right|^2}{\mu_n^2} = 1.$$

□

As a direct consequence of Theorem 4.24 and the Arzelà-Ascoli theorem, we get for the generated manifolds M:

Corollary 4.25. *Let $(p_n, t_n) \in S^1 \times [0, T)$ and thus $(p_n, 0, t_n) \in S^1 \times S^1 \times [0, T)$ be an essential blow-up sequence. Then there exists a subsequence of (p_n, t_n) along which the rescaled solutions of the generated surfaces M_n converge in $C^\infty(K)$ to a smooth,* *nontrivial limit M_∞, for every compact set $K \subset \mathbb{R}^2 \times (-\infty, 0]$. The limit M_∞ is a solution to the mean curvature flow and it exists at least on the time interval $(-\infty, 0]$. The limit is a cylinder in $\mathbb{R}^4$ generated by c_∞:*

$$M_\infty := F_\infty(\mathbb{R} \times \mathbb{R}) := \left\{ \left(c_\infty^1(x), c_\infty^2(x), c_\infty^3(x), y^4 \right), x \in \mathbb{R}, y^4 \in \mathbb{R} \right\}.$$

Especially, the mean curvature vector of M_∞ and c_∞ coincide.

Since the limiting family c_∞ is a solution to the mean curvature flow, Altschuler's result proved in [4, Theorem 7.7] apply here, too:

Corollary 4.26. *c_∞ is a family of convex planar curves.*

From Lemma 3.6, it follows immediately that limiting family is locally planar in all points where the curvature does not vanish. But it may change the plane in points $H = 0$, such that it may not be globally planar. To proof that this cannot happen, it is necessary to know that a solution of the curve shortening flow is analytic in space for all positive times $t > 0$.

For plane initial curves $c_0 \in C^{2,\alpha}\left(S^1\right)$ the analycity in space for $t > 0$ follows immediately from the theory of parabolic equations (see [7], [21]). Also if the initial curve is only piecewise C^1 and planar, analycity can be shown under certain growing conditions [31, Theorem 1.2]. Similar results were proved in [48], where the authors showed that there exists a global unique and analytic solution to the mean curvature flow of entire graphs for Lipschitz initial data with small Lipschitz norm.

But there are more general results on the regularity of solutions of parabolic differential equations. Escher and Simonett proved analycity in space and time even for solutions of fully nonlinear parabolic evolution equations on symmetric spaces. Since the Laplace-Beltrami-operator is analytic and invariant under the transformation group of S^1 their result [26, Theorem 1.1] is applicable to the curve shortening flow for space curves and hence, a solutions $c : S^1 \times [0, T) \to \mathbb{R}^m$ to the curve shortening flow with smooth initial data c_0 is real analytic in space and time for every positive time $t > 0$.

Proposition 4.27. *Suppose $c : S^1 \to \mathbb{R}^3$ is a regular parametrized, analytic curve with isolated inflection points $H = 0$. Furthermore, assume that the torsion vanishes whenever the curvature does not. Then c is globally planar.*

Proof. Lemma 3.6 implies that c has to be planar on every subarc where $H \neq 0$. Assume c is not globally planar. Since there exist only isolated inflection points, c has to change the plane in a point x_0 where $H(x_0) = 0$. But in such a point x_0, all higher derivatives of H have to vanish $\nabla^N H = 0$, which contradicts the condition that c and thus, H is analytic. □

Now for completeness, the proof of Corollary 4.26 will be given.

Proof of Corollary 4.26. Let (p_n, t_n) be an essential blow-up sequence such that the rescaled solutions converge smoothly to c_∞. Because the integral is invariant under rescaling, the integral estimate Corollary 4.19 implies that for all $\epsilon > 0$ there exists a $\theta > 0$ sufficiently small such that for all times t_n large enough

$$\epsilon > \int_{t_n-\theta}^{t_n} \int_c |H| \, |\tau|^2 \, \mathrm{d}\mu \mathrm{dt} = \int_{-\theta\mu_n^2}^{0} \int_{c_n} |H_n| \, |\tau_n|^2 \mathrm{d}\mu_n \mathrm{d}\bar{t},$$

where τ_n denotes the torsion of the rescaled curve c_n, i. e. $\int_{-\infty}^{0} \int_{c_\infty} |H_\infty| \, |\tau_\infty|^2 \mathrm{d}\mu_\infty \mathrm{d}\bar{t} = 0$. Hence, the torsion has to vanish whenever the mean curvature H_∞ does not.

If $H_\infty (x, t) \neq 0$ and $\tau (x, t) = 0$ for $x \in I \subset \mathbb{R}$, I an interval, it is a basic fact that $c_\infty (x, t)$ is planar on I. But if $H_\infty = 0$ at an isolated point, c_∞ has not to be planar. However, c_∞ is analytic and therefore globally planar as shown in Proposition 4.27.

To prove the convexity, first note that as above the rescaling procedure implies

$$\frac{\partial}{\partial \bar{t}} \int_{c_\infty} |H_\infty| \, \mathrm{d}\mu_\infty = 0.$$

Secondly, c_∞ is a family of planar solutions to the mean curvature flow, hence, one can define its curvature k_∞ (see [60]). Proposition 4.28 below gives:

$$\int_{-\infty}^{0} \sum_{p; k_\infty(p,\cdot)=0} \left| \frac{\partial k_\infty}{\partial s} \right| \mathrm{d}\bar{t} = 0,$$

where s is the parametrization of c_∞ by arc length. Hence, every inflection point of c_∞ must be degenerate (i. e. $k_\infty = \frac{\partial k_\infty}{\partial s} = 0$). Angenent's result Theorem 1.4 in [8] translated to this situation yields: Since c_∞ is a non trivial planar solution to the mean curvature flow, the number of inflection points is finite for each time t_1, non increasing in time and it decreases on degenerate inflection points. Therefore, if there would be a time t_1 at which c_∞ has a inflection point $k_\infty (p_1, t_1) = 0$, it automatically had to have infinite many degenerate inflections points, hence it would be a line. Since $k_\infty (0, 0) = 1$ there cannot be any inflection points, so c_∞ is a family of convex curves. □

The idea of the following proposition goes back to [31]. Since it is only stated in [4], the proof is given for completeness.

Proposition 4.28. *Let γ be a planar solution to the mean curvature flow, and k be the curvature of γ. The time derivative of the total curvature is given by*

$$\frac{\partial}{\partial t} \int_\gamma |k| \; ds = -2 \sum_{p; k(p,\cdot)=0} \left| \frac{\partial k}{\partial s} \right|,$$

where s denotes the parametrization by arc length.

Proof. Let s be the parametrization of γ by arc length. Then the evolution equation for the curvature is $\frac{\partial}{\partial t} k = \frac{\partial^2 k}{\partial s^2} + k^3$ (see for example [29]). Define $I (t) := \{p; k (p, t) > 0\}$.

Since γ_t is analytic for $t > 0$, its curvature has only finite many zeros, and $I(t)$ is a countable union of intervals $I(t) = \bigcup_i (a_i, b_i)$. Splitting the integral leads to:

$$\int_\gamma |k|\,\mathrm{d}s = \int_{I(t)} k\mathrm{d}s - \int_{\gamma\setminus I(t)} k\mathrm{d}s = \sum \left(\int_{a_i}^{b_i} k\mathrm{d}s - \int_{b_i}^{a_{i+1}} k\mathrm{d}s \right).$$

Using $\frac{\partial}{\partial t}\mathrm{d}s = -k^2\mathrm{d}s$, differentiation to t gives

$$\frac{\partial}{\partial t}\left(\sum \left(\int_{a_i}^{b_i} k\mathrm{d}s - \int_{b_i}^{a_{i+1}} k\mathrm{d}s\mathrm{d}\mu \right)\right) = \sum \left(\int_{a_i}^{b_i} \frac{\partial^2 k}{\partial s^2}\mathrm{d}s - \int_{b_i}^{a_{i+1}} \frac{\partial^2 k}{\partial s^2}\mathrm{d}s \right).$$

Here the boundary terms vanish by definition of $I(t)$. Integration by parts yields

$$\frac{\partial}{\partial t}\left(\int_{I(t)} k\mathrm{d}\mu - \int_{\gamma\setminus I(t)} k\mathrm{d}\mu \right) = \sum 2 \left(\frac{\partial k}{\partial s}(b_i) - \frac{\partial k}{\partial s}(a_i) \right) = -2 \sum_{p;H(p,\cdot)=0} \left| \frac{\partial k}{\partial s} \right|$$

□

4.3.2. Behaviour in type-I and type-II singularities

Now, the limiting behaviour of rescaled solutions to the perturbed flow will be studied depending on which type of singularity occurs. If condition (4.2) is satisfied, the perturbed flow looks like the mean curvature flow in the singularity (cp. [4]).

Definition 4.29. *$p_0 \in \mathbb{R}^3$ is a* blow-up point for the perturbed flow *if there exists a point $p \in S^1$ such that $c(p,t) \to p_0$ for $t \to T$ (and hence $F(p,0,t) \to (p_0,0)$).*

First assume, the occurring singularity is of type-I. The *rescaled solutions* in p_0 of the generated surfaces and the corresponding curves are defined via

$$\tilde{F}\left(p,\phi,\tilde{t}\right) := (2(T-t))^{-\frac{1}{2}} \left(F(p,\phi,t) - (p_0,0)\right) \tag{4.27}$$

$$\tilde{c}\left(p,\tilde{t}\right) := (2(T-t))^{-\frac{1}{2}} \left(c(p,t) - p_0\right), \text{ where } \tilde{t} := -\frac{1}{2}\log(T-t). \tag{4.28}$$

So the rescaled solutions are defined on the time interval $\left[-\frac{1}{2}\log T, \infty\right)$.

One checks that

$$\frac{\partial}{\partial \tilde{t}}\tilde{F} = \tilde{F} + \tilde{\mathcal{H}}, \quad \frac{\partial}{\partial \tilde{t}}\tilde{c} = \tilde{c} + \tilde{H} + \tilde{\lambda} \tag{4.29}$$

where $\tilde{\mathcal{H}} = (2(T-t))^{\frac{1}{2}}\mathcal{H}$ denotes the mean curvature vector of the rescaled surface $\tilde{M}$, $\tilde{H}$ is the mean curvature vector of $\tilde{c}$, and $\tilde{\lambda}$ is defined such that $\tilde{\mathcal{H}} = \tilde{H} + \tilde{\lambda}$. This is

$$\tilde{H} = (2(T-t))^{\frac{1}{2}} H, \quad \tilde{\lambda} := \frac{-1}{\tilde{r} + (2(T-t))^{-\frac{1}{2}} r_p} e_1^\perp, \quad \tilde{r} = (2(T-t))^{-\frac{1}{2}}(r - r_p),$$

with $\tilde{r} := \langle \tilde{c}, e_1 \rangle$ and $r_p := \langle p_0, e_1 \rangle$. Note again that in general $\tilde{\lambda} \neq -\frac{1}{\tilde{r}} e_1^{\perp}$. If the forming singularity is of type-I, the curvatures of the rescaled solutions are bounded for all times: $\left|\tilde{\mathcal{H}}\right| \leqslant C$ and $\left|\tilde{H}\right| \leqslant C$ for some $C < \infty$.

To handle type-I singularities it is a frequent practice to use Huisken's monotonicity formula for mean curvature flow (see Lemma 4.31 proved in [38]. Here it will be used for the generated surfaces M. Recall the definition of the backwards heat kernel:

Definition 4.30. *Let $\rho_{p_0}(x,t)$ be the* backwards heat kernel *flowing out of a point $(p_0, T) \in \mathbb{R}^m \times [0,T]$, i. e.*

$$\rho_{p_0}(x,t) := \frac{1}{(4\pi(T-t))^{\frac{m}{2}}} \exp\left(-\frac{|x-p_0|^2}{4(T-t)}\right), \quad x \in \mathbb{R}^m,\ t < T. \tag{4.30}$$

The modified heat kernel *is defined by*

$$\tilde{\rho}(x) = \exp\left(-\frac{1}{2}|x|^2\right), \quad x \in \mathbb{R}^m. \tag{4.31}$$

For the surfaces $M_t \subset \mathbb{R}^4$ Huisken's monotonicity formula reads as follows:

Lemma 4.31. *The monotonicity formulas for the surfaces M_t, $t \in [0,T)$, and the rescaled surfaces $\tilde{M}_{\tilde{t}}$, $\tilde{t} \in \left[-\frac{1}{2}\log T, \infty\right)$, are given by*

$$\frac{\partial}{\partial t}\int_M \rho_{p_0} d\eta = -\int_M \rho_{p_0}\left|\mathcal{H} + \frac{(F-p_0)^{\perp}}{2(T-t)}\right|^2 d\eta, \tag{4.32}$$

$$\frac{\partial}{\partial \tilde{t}}\int_{\tilde{M}} \tilde{\rho} d\tilde{\eta} = -\int_{\tilde{M}} \tilde{\rho}\left|\tilde{\mathcal{H}} + \tilde{F}^{\perp}\right|^2 d\tilde{\eta}, \tag{4.33}$$

where $^{\perp}$ denotes the normal part of the vector.

Integrating the monotonicity formula in time gives:

Corollary 4.32. *For all $\epsilon > 0$, there exists a time $\theta < \infty$ such that*

$$\int_{\theta}^{\infty}\int_{\tilde{M}} \left|\tilde{\mathcal{H}} + \tilde{F}^{\perp}\right|^2 d\tilde{\eta} d\tilde{t} < \epsilon. \tag{4.34}$$

Let $\tilde{F}_n := \tilde{F}\left(\cdot, \tilde{t}_n\right)$, $\tilde{c}_n := \tilde{c}\left(\cdot, \tilde{t}_n\right)$, for some time $-\log T < \tilde{t}_n < \infty$. Now, Corollary 4.32 implies

Proposition 4.33. *Let p_0 be a blow-up point such that $\lim_{n\to\infty} F(p,0,t_n) = (p_0, 0)$. Then the rescalings $\tilde{M}_n$ of M (resp. $\tilde{c}_n$ of c) converge along a subsequence to a nonempty limiting solution $\tilde{M}_{\infty}$ (resp. $\tilde{c}_{\infty}$) moving by homothety. Furthermore, $\tilde{c}_{\infty}$ is a family of convex, planar curves and each limit $\tilde{c}_{\infty}$ has the same winding number.*

Proof. Since the singularity is of type-I, there is a point p such that $\tilde{F}\left(p,0,\tilde{t}\right)$ remains bounded as $\tilde{t} \to \infty$. Moreover, there exists by definition a constant c_1 such that $\mathcal{M}_t\left(T-t\right) \leqslant c_1$, for all $t \in [0,T)$. On the other hand, (2.5) implies $\frac{1}{\sigma(T-t)} \leqslant \mathcal{M}_t$, for all $t \in [0,T)$ (cp. [38]). Hence in the type-I case, each blow-up sequence is essential.

Furthermore, $\left|\tilde{\mathcal{H}}\right|^2$, $\left|\tilde{H}\right|^2$, $\left|\tilde{\lambda}\right|^2$ are bounded along every sequence $\tilde{t}_n \to \infty$. Hence, the proof of convergence of $\tilde{c}_n$ along a subsequence is the same as above in the proof of Theorem 4.24, and the convergence of $\tilde{c}_n$ induces convergence of $\tilde{M}_n$ to some limit $\tilde{M}_\infty$ generated by $\tilde{c}_\infty$. Since the limit $\tilde{M}_\infty$ is not empty, (4.34) implies

$$\left|\tilde{\mathcal{H}}_\infty + \tilde{F}_\infty^\perp\right|^2 = 0.$$

As above, $\lambda_\infty = 0$ and especially, $\left|\tilde{H}_\infty + \tilde{c}_\infty^\perp\right|^2 = 0$, i. e. $\tilde{c}_\infty$ is one of the Abresch-Langer solutions (cp. [1]).

As in the proof of Corollary 4.26, the limit $\tilde{c}_\infty$ is a family of convex, planar curves. Therefore, $\frac{1}{2\pi}\int_{\tilde{c}_\infty}\left|\tilde{H}_\infty\right| \mathrm{d}\tilde{\mu}_\infty$ is the winding number of $\tilde{c}_\infty$ and it suffices to show that the limit of $\int_{\tilde{c}_n}\left|\tilde{H}\right| \mathrm{d}\tilde{\mu}$ is finite and unique. Since the limit $\lim_{t\to T}\int_c |H|\, \mathrm{d}\mu < \infty$ exists (cp. the proof of Lemma 4.17), the limit of the scaling invariant total curvature $\int_{\tilde{c}_n}\left|\tilde{H}\right| \mathrm{d}\tilde{\mu} = \int_c |H|\, \mathrm{d}\mu\left(t_n\right)$ is unique. Hence, each limit $\tilde{c}_\infty$ has the same finite winding number, especially the limits cannot be Abresch-Langer solutions with infinite many loops, and the limits differ only by dilation and a euclidean motion. □

The behaviour of curves evolving under mean curvature flow in type-II singularities was characterized by Altschuler. Since it was shown in Theorem 4.24 that the limit c_∞ of the rescaled solutions c_n evolves under mean curvature flow, Altschuler's Theorem 8.16 in [4] applies to c_∞, too:

Lemma 4.34. *If the occurring singularity is of type-II, there exists an essential blow-up sequence (p_n, t_n) such that a limit of rescalings c_n along (p_n, t_n) converges to the Grim Reaper.*

The *Grim Reaper* was first mentioned in in [29]. It is a translating solution to the mean curvature flow (see Figure 4.6.) given by $c_t = \operatorname{graph}\left(u\left(x,t\right)\right)$ with

$$u\left(x,t\right) = -\log\cos\left(x\right) + t.$$

The proof of Lemma 4.34 is divided into two steps. First Altschuler showed that in the case of a type-II singularity, there exists an essential blow-up sequence along which the rescaled solutions converge to a family of solutions whose curvature is bounded for all times $t \in (-\infty, \infty)$. The second step is to proof that this limit is the Grim Reaper. Therefore, it is observed that since the limiting curve is planar and convex, it has to turn exactly π and is embedded. The maximum-principle is used to proof that in well chosen coordinates[1] the curvature of the limiting curve does not change in time. Thus,

[1] The curve is parametrized by (θ, τ), where θ is the angle between the tangent vector and the y^1-axis, τ is the time parameter with fixed θ.

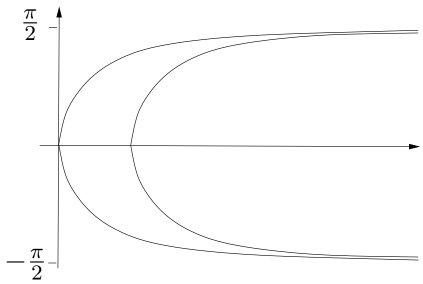

Figure 4.6.: The Grim Reaper

it is a planar, convex, translating solution with total curvature π, such a curve is the Grim Reaper.

5. Notes on singularities on the plane of rotation

In Chapter 4, it was a necessary condition that the distance of the curve to the plane of rotation was bounded away from zero $r \geqslant r_0$ on $[0, T)$. In this case, the curvature vector λ coming from the rotation keeps bounded in the singularity, and only the mean curvature of the curve blows up, so the behaviour of the perturbed flow in the singularity is the same as of the mean curvature flow.

Now, the behaviour of the perturbed flow is investigated in singularities in which both, the mean curvature of the curve and the curvature coming from the rotation, blow up. An example for this type of singularity was given in in the example above Remark 4.2.

5.1. A monotonicity formula

Let $c : S^1 \times [0, T) \to \mathbb{R}^3_{>0}$ be a solution to the perturbed flow

$$\frac{\partial}{\partial t} c(x, t) = H(x, t) + \lambda(x, t), \tag{5.1a}$$

$$c(x, 0) = c_0(x), \; c(x, t) \in \mathbb{R}^3_{>0} \tag{5.1b}$$

on the maximal time interval of existence $[0, T)$.

Let $p_0 \in \mathbb{R}^3_{\geqslant 0}$ be a blow-up point. As above, the rescaled solutions of the curves in p_0 are defined via

$$\tilde{c}\left(p, \tilde{t}\right) := \left(2\left(T - t\right)\right)^{-\frac{1}{2}} \left(c\left(p, t\right) - p_0\right), \text{ where } \tilde{t} := -\frac{1}{2} \log\left(T - t\right). \tag{5.2}$$

So, the rescaled solutions are defined on the time interval $\left[-\frac{1}{2}\log T, \infty\right)$ and evolve like

$$\frac{\partial}{\partial \tilde{t}} \tilde{c} = \tilde{c} + \tilde{H} + \tilde{\lambda}$$

with

$$\tilde{H} = \left(2\left(T - t\right)\right)^{\frac{1}{2}} H, \quad \tilde{\lambda} := \frac{-1}{\tilde{r} + \left(2\left(T - t\right)\right)^{-\frac{1}{2}} r_p} e_1^{\perp}, \quad \tilde{r} = \left(2\left(T - t\right)\right)^{-\frac{1}{2}} \left(r - r_p\right),$$

where $\tilde{r} := \langle \tilde{c}, e_1 \rangle$ and $r_p := \langle p_0, e_1 \rangle$.

Using the backward heat kernel in $\mathbb{R}^3$ (cp. Definition 4.30), the following monotonicity formula is valid for the perturbed flow.

Lemma 5.1. *Let $p_0 \in \mathbb{R}^3_{\geqslant 0}$, $r_p = \langle p_0, e_1\rangle$ and ρ_{p_0} (resp. $\tilde{\rho}$) be the corresponding (modified) heat kernel. For $t \in [0,T)$, $\tilde{t} \in \left[-\frac{1}{2}\log\left(T-t\right), \infty\right)$, we have:*

$$\frac{\partial}{\partial t}\int_c \frac{r\rho_{p_0}}{\sqrt{2\left(T-t\right)}}\, d\mu = -\int_c \frac{r\rho_{p_0}}{\sqrt{2\left(T-t\right)}}\left|H+\lambda+\frac{\left(c-p_0\right)^{\perp}}{2\left(T-t\right)}\right|^2 d\mu + r_p\int_c \frac{\rho_{p_0}}{\sqrt{2\left(T-t\right)}^3}\, d\mu \tag{5.3}$$

and

$$\frac{\partial}{\partial \tilde{t}}\int_{\tilde{c}}\left(\tilde{r}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\, d\tilde{\mu} = -\int_{\tilde{c}}\left(\tilde{r}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\left(\left|\tilde{H}+\tilde{\lambda}+\tilde{c}^{\perp}\right|^2\right)d\tilde{\mu}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\int_{\tilde{c}}\tilde{\rho}\, d\tilde{\mu}. \tag{5.4}$$

If $r_p = 0$, i. e. the singularity occurs on the plane of rotation, these equations give monotonicity formulas for the (rescaled) perturbed flow.

Proof. The time derivative of the backward heat kernel is given by

$$\frac{\partial}{\partial t}\rho_{p_0}\left(c,t\right) = \rho_{p_0}\left(\frac{1}{2\left(T-t\right)}-\frac{\left|c-p_0\right|^2}{4\left(T-t\right)^2}-\frac{\left\langle \frac{\partial}{\partial t}c, c-p_0\right\rangle}{2\left(T-t\right)}\right), \tag{5.5}$$

and hence (3.5) and (3.7) imply

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_c \frac{r}{\sqrt{2\left(T-t\right)}}\rho_{p_0}\,\mathrm{d}\mu \\
&= -\int_c \frac{r}{\sqrt{2\left(T-t\right)}}\rho_{p_0}\left(-\frac{1}{2\left(T-t\right)}+\frac{\left|c-p_0\right|^2}{4\left(T-t\right)^2}+\frac{\left\langle H+\lambda, c-p_0\right\rangle}{2\left(T-t\right)}\right)\mathrm{d}\mu \\
&\quad +\int_c \frac{\left(\Delta r - r\left|\lambda\right|^2\right)}{\sqrt{2\left(T-t\right)}}\rho_{p_0}-\frac{r}{\sqrt{2\left(T-t\right)}}\rho_{p_0}\left(\left|H\right|^2+\left\langle H,\lambda\right\rangle\right)\mathrm{d}\mu+\int_c \frac{r}{\sqrt{2\left(T-t\right)}^3}\rho_{p_0}.
\end{aligned}$$

Using $\Delta r = -r\left\langle H,\lambda\right\rangle$ we get

$$\begin{aligned}
\frac{\partial}{\partial t}\int_c \frac{r}{\sqrt{2\left(T-t\right)}}\rho_{p_0}\,\mathrm{d}\mu &= -\int_c \frac{r}{\sqrt{2\left(T-t\right)}}\rho_{p_0}\left(\left|H+\lambda+\frac{c-p_0}{2\left(T-t\right)}\right|^2\right)\mathrm{d}\mu \\
&\quad +\int_c \frac{r\rho_{p_0}}{\sqrt{2\left(T-t\right)}^3}\left(2+\left\langle H+\lambda, c-p_0\right\rangle\right)\mathrm{d}\mu.
\end{aligned} \tag{5.6}$$

For an arbitrary differentiable vector field $Y : c \to \mathbb{R}^3$, the divergence with respect to c is defined by $div\, Y = g^{ij}\delta_{\alpha\beta}c_i^{\alpha}\frac{\partial Y^{\beta}}{\partial x^j}$ (see for example [25]). If Y is normal, this implies

$$div\, Y = -g^{ij}\delta_{\alpha\beta}\frac{\partial^2 c^{\alpha}}{\partial x^i \partial x^j}Y^{\beta} = -\left\langle H, Y\right\rangle.$$

Hence, for a differentiable vector field Y in $\mathbb{R}^3$, the divergence theorem yields:

$$\int_M div\, Y \mathrm{d}\mu = \int_M div\, Y^\top + div\, Y^\perp \mathrm{d}\mu = -\int_M \langle H, Y\rangle \,\mathrm{d}\mu,$$

where $^\top$ denotes the orthonormal projection onto the tangent bundle, and $Y^\perp$ is the normal part of Y. For $Y = \frac{r\rho_{p_0}}{\sqrt{2(T-t)}^3}(c-p_0)$ one computes

$$\begin{aligned}
div\left(\frac{r\rho_{p_0}}{\sqrt{2(T-t)}^3}(c-p_0)\right) =& \frac{r\rho_{p_0}}{\sqrt{2(T-t)}^3} + \frac{\rho_{p_0}}{\sqrt{2(T-t)}^3}\delta_{\alpha\beta}\nabla^m r c_m^\alpha (c-p_0)^\beta \\
&- \frac{r\rho_{p_0}}{\sqrt{2(T-t)}^5}\left(\delta_{\alpha\beta}c_i^\alpha (c-p_0)^\beta g^{ij}\delta_{\gamma\delta}c_i^\gamma (c-p_0)^\delta\right) \\
=& \frac{\rho_{p_0}}{\sqrt{2(T-t)}^3}\left(r - r\frac{\left|(c-p_0)^\top\right|^2}{2(T-t)} + \left\langle e_1^\top, c-p_0\right\rangle\right),
\end{aligned}$$

where $\nabla_n r g^{mn} c_m^\alpha = \delta_{\gamma\delta}c_n^\gamma e_1^\delta g^{mn} c_m^\alpha = e_1^\top$ is used. Using the divergence theorem and $\lambda = -\frac{1}{r}e_1^\perp$, (5.6) becomes

$$\begin{aligned}
\frac{\partial}{\partial t}\int_c \frac{r}{\sqrt{2(T-t)}}\rho_{p_0}\mathrm{d}\mu =& -\int_c \frac{r}{\sqrt{2(T-t)}}\rho_{p_0}\left(\left|H + \lambda + \frac{(c-p_0)^\perp}{2(T-t)}\right|^2\right)\mathrm{d}\mu \\
&+ \int_c \frac{\rho_{p_0}}{\sqrt{2(T-t)}^3}\left(r - \left\langle e_1^\top, c-p_0\right\rangle - \left\langle e_1^\perp, c-p_0\right\rangle\right)\mathrm{d}\mu
\end{aligned}$$

Here, the terms in the bracket in the second integral add up to r_p.

For the rescaled solution one easily checks

$$\begin{aligned}
\frac{\partial}{\partial \tilde{t}}\tilde{r} &= \left\langle \tilde{H}, e_1\right\rangle + \left\langle \tilde{\lambda}, e_1\right\rangle + \tilde{r} = -\left(\tilde{r} + \frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\left(\left\langle \tilde{H}, \tilde{\lambda}\right\rangle + \left|\tilde{\lambda}\right|^2\right) + \tilde{r} \\
\frac{\partial}{\partial \tilde{t}}\mathrm{d}\tilde{\mu} &= \left(1 - \left(\left|\tilde{H}\right|^2 + \left\langle \tilde{H}, \tilde{\lambda}\right\rangle\right)\right)\mathrm{d}\tilde{\mu},
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial}{\partial \tilde{t}}\int_{\tilde{c}}\left(\tilde{r} + \frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\mathrm{d}\tilde{\mu} =& -\int_{\tilde{c}}\left(\tilde{r} + \frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\left(|\tilde{c}|^2 + \left\langle \tilde{c}, \tilde{H}\right\rangle + \left\langle \tilde{c}, \tilde{\lambda}\right\rangle\right)\mathrm{d}\tilde{\mu} \\
&- \int_{\tilde{c}}\left(\tilde{r} + \frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\left(\left|\tilde{H}\right|^2 + \left\langle \tilde{H}, \tilde{\lambda}\right\rangle - 1\right)\mathrm{d}\tilde{\mu} \\
&- \int_{\tilde{c}}\tilde{\rho}\left(\tilde{r} + \frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\left(\left\langle \tilde{H}, \tilde{\lambda}\right\rangle + \left|\tilde{\lambda}\right|^2 - 1\right)\mathrm{d}\tilde{\mu}.
\end{aligned}$$

As above the divergence theorem and

$$div\left(\left(\tilde{r}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\tilde{c}\right)=\left(\tilde{r}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}-\left(\tilde{r}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\left|\tilde{c}^{\top}\right|^2+\tilde{\rho}\left\langle e_1^{\top},\tilde{c}\right\rangle$$

yield

$$\begin{aligned}\frac{\partial}{\partial\tilde{t}}\int_{\tilde{c}}\left(\tilde{r}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\mathrm{d}\tilde{\mu}=&-\int_{\tilde{c}}\left(\tilde{r}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\left(\left|\tilde{H}+\tilde{\lambda}+\tilde{c}^{\perp}\right|^2\right)\mathrm{d}\tilde{\mu}\\&+\int_{\tilde{c}}\left(\tilde{r}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\left(1+\left\langle\tilde{\lambda},\tilde{c}\right\rangle-\left(\tilde{r}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)^{-1}\left\langle e_1^{\top},\tilde{c}\right\rangle\right)\\=&-\int_{\tilde{c}}\left(\tilde{r}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\right)\tilde{\rho}\left(\left|\tilde{H}+\tilde{\lambda}+\tilde{c}^{\perp}\right|^2\right)\mathrm{d}\tilde{\mu}+\frac{e^{\tilde{t}}}{\sqrt{2}}r_p\int_{\tilde{c}}\tilde{\rho}\mathrm{d}\tilde{\mu}\end{aligned}$$

□

In the previous proof, it is essential not to integrate ρ_{p_0}, as in the case of the mean curvature flow, but to multiply by r. Otherwise the terms coming from the perturbation of the flow would not cancel out in this nice way. Moreover, we do not expect to get a better formula.

Integrating the monotonicity formula in time gives:

Corollary 5.2. *Let p_0 be a blow-up point with $r_{p_0}=\langle p_0,e_1\rangle=0$. Then for all $\epsilon>0$, there exists a time $\vartheta<\infty$ such that*

$$\int_{\vartheta}^{\infty}\int_{\tilde{c}}\tilde{r}\tilde{\rho}\left(\left|\tilde{H}+\tilde{\lambda}+\tilde{c}^{\perp}\right|^2\right)d\tilde{\mu}d\tilde{t}<\epsilon. \tag{5.7}$$

As in [38], this implies

Theorem 5.3. *Assume that $\left(\frac{C_0}{T-t}\leqslant\right)\max_t\left(\frac{1}{r^2}\right)\leqslant\frac{C_1}{T-t}$ and $\left(\frac{C_2}{T-t}\leqslant\right)\max_t|H|^2\leqslant\frac{C_3}{T-t}$ for some $C_0,\dots,C_3>0$. Let p_0 be a blow-up point with $r_p=\langle e_1,p_0\rangle=0$ such that there exists a point p and a sequence $\{t_n\}$ with*

$$\lim_{n\to\infty}c(p,t_n)=p_0,\quad\lim_{n\to\infty}|H(p,t_n)|^2=\infty,\quad\lim_{n\to\infty}|\lambda(p,t_n)|^2=\infty,$$

i. e. the occurring singularity is of type-I and both, the curvature of the curve and the curvature coming from the rotation, blow up. Then for every compact set $K\subset\mathbb{R}$ there exists a subsequence of $\{t_n\}$ along which the rescalings $\tilde{c}_n=\tilde{c}\left(\cdot,\tilde{t}_n\right)$ considered as periodic curves on $\mathbb{R}$ converge in $C^{\infty}(K)$ to some limit $\tilde{c}_\infty$ with $\tilde{c}_\infty^{\perp}=-\tilde{H}_\infty-\tilde{\lambda}_\infty$.

Proof. Note that $\tilde{c}_n(p)=\tilde{c}\left(p,\tilde{t}_n\right)$ remains bounded as $n\to\infty$. Furthermore, in this setting as in the proof of Proposition 4.33, every blow-up sequence (p,t_n) is essential.

Since $\left|\tilde{\mathcal{H}}\right|$, $\left|\tilde{H}\right|$ and $\left|\tilde{\lambda}\right|$ are bounded the proof of convergence is exactly the same as in Theorem 4.24, but for convenience the proof will be outlined. Again, think of $\tilde{c}$ as periodic curves $\tilde{c}_n : \mathbb{R} \to \mathbb{R}^3_{\geqslant 0}$ with parametrization u such that

$$\frac{\partial}{\partial u}\tilde{c}_n = \sqrt{2\left(T-t_n\right)}\frac{\partial}{\partial x}\tilde{c}_n\left(=\frac{\partial}{\partial x}c\right).$$

Since (p,t_n) is an essential blow-up sequence, all higher derivatives $\left|\nabla^N \tilde{H}_n\left(u\right)\right|^2$ are bounded (cp. (4.26)):

$$\begin{aligned}\rho^{N+1}\left|\nabla^N \tilde{H}_n\left(u\right)\right|^2 &= \rho^{N+1}\left(2\left(T-t_n\right)\right)^{N+1}\left|\nabla^N H\left(u,t_n\right)\right|^2\\ &\leqslant \tilde{c_N}\rho^{N+1}\left(2\left(T-t_n\right)\right)^{N+1}\mathcal{M}_s{}^{N+1} \leqslant \mathsf{d}_N\left(2\left(T-t_n\right)\right)^{N+1}\left|H\left(p,t_n\right)\right|^{2N+2} \leqslant c_n, \quad (5.8)\end{aligned}$$

where the first inequality is comes from Proposition 4.5, the second inequality is due to the fact that the curvature of the curves blows up with the same rate as $|\lambda|$ and the third inequality is valid by definition.

Hence, the Arzelà-Ascoli theorem implies the convergence of $T_n := \frac{\partial}{\partial u}\tilde{c}$ along a subsequence (p,t_{n_k}) to a smooth limit T_∞. Integration gives a smooth, non empty limit $\tilde{c}_\infty$. Since $C_1 \leqslant \max \tilde{H}_n \leqslant C_2$, the limit is not trivial.

Remember the evolution equation of r (cp. (3.7))

$$\frac{\partial}{\partial t}r = \Delta r - r\left|\lambda\right|^2 \geqslant \Delta r - \frac{1}{r}.$$

The maximum principle implies

$$\min r\left(s\right)^2 - \min r\left(t\right)^2 \geqslant -2\left(s-t\right)$$

and thus, if $\min r\left(s\right) \to 0$ as $s \to T$,

$$\max_t\left(\frac{1}{r^2}\right) \geqslant \frac{C_0}{2\left(T-t\right)}$$

and therefore, $\tilde{\lambda}_\infty$ is not trivial. Furthermore, the assumption $\max_t\left(\frac{1}{r^2}\right) \leqslant \frac{C_1}{T-t}$ leads to $\inf \tilde{r}_\infty > 0$, so that $\tilde{r}_\infty$ does not vanish. Corollary 5.2 now implies

$$\left|\tilde{c}_\infty^\perp + \tilde{H}_\infty + \tilde{\lambda}_\infty\right|^2 = 0.$$

□

Remark 5.4. Let c_0 be a solution to $c_0 = -H_0 - \lambda_0$. Then the generated surface $F\left(x,\phi\right) = \left(c\left(x\right),\phi\right)$ is a self-shrinker to the mean curvature flow, in cylindrical coordinates (cp. Section B.1) one computes

$$\left(F^\perp\left(x,\phi\right)\right)^\alpha\frac{\partial}{\partial z^\alpha} = \left(F^\alpha - h_{\beta\gamma}F^\beta F^\gamma_m \bar{g}^{mn}F^\alpha_n\right)\frac{\partial}{\partial z^\alpha} = \left(c^\perp,0\right) = -\left(H+\lambda,0\right) = -\mathcal{H}$$

Hence, each self-shrinker to the perturbed flow leads to a self-shrinker of the mean curvature flow and vice versa.

Theorem 5.3 can be reformulated in terms of the surface. Therefore, recall the rescaling of M (see (4.27)):

$$\tilde{F}\left(p, \phi, \tilde{t}\right) := \left(2\left(T-t\right)\right)^{-\frac{1}{2}}\left(F\left(p, \phi, t\right) - \left(p_0, 0\right)\right), \quad \tilde{t} := -\frac{1}{2}\log\left(T-t\right)$$

Note that since here $r_p := \langle e_1, p_0\rangle = 0$, the plane of rotation is fixed under the rescaling.

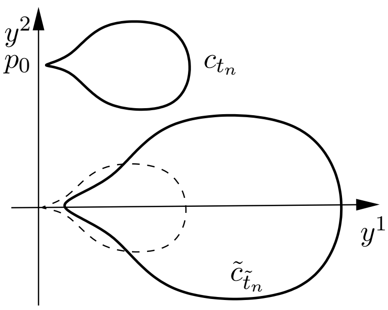

Figure 5.1.: Rescaling does not change the plane of rotation.

Corollary 5.5. *Let $F : S^1 \times S^1 \times [0, T) \to \mathbb{R}^4$ be a rotationally symmetric solution to the mean curvature flow such that the conditions of Theorem 5.3 are satisfied. Then the rescalings $\tilde{F}\left(\cdot, t_n\right)$ converge along a subsequence to some limit $\tilde{F}_\infty$ with $\tilde{F}_\infty^\perp = -\mathcal{H}$.*

5.1.1. Notes on self-similar solutions to the perturbed flow

Proposition 5.6. *Let $\gamma : S^1 \to \mathbb{R}^2$ be one of the Abresch-Langer curves, then $c : S^1 \to \mathbb{R}^3_{>0}$ defined as $c_0\left(\cdot\right) := \left(1, \gamma\left(\cdot\right)\right)$ is a self-similar solution to the perturbed flow, thus $c\left(\cdot, t\right) = \sqrt{1-2t}c\left(\cdot\right)$ is a self-shrinker satisfying the conditions of Theorem 5.3.*

Proof. By definition of c, its curvature vector H and the perturbation vector λ are given by

$$H = \left(0, H_\gamma\right), \quad \lambda = -\frac{1}{r}e_1^\perp = -e_1.$$

Hence, $c^\perp = -H - \lambda$ follows since γ is an Abresch-Langer curve. □

Since the self-shrinkers to the curve shortening flow, i. e. the Abresch-Langer curves, are convex, it is a natural question to ask, if a similar condition is satisfied for planar self-shrinkers to the perturbed flow, too.

Remark 5.7. Let c be a solution to $c^{\perp} = -H - \lambda$, with $H \neq 0$ and $\tau = 0$, i. e. c lies in a plane $E^2 \subset \mathbb{R}^3$. Then E contains the first coordinate vector field e_1 of $\mathbb{R}^3$ or $e_1 \perp E$.

This follows easily from differentiating the equality $c^{\perp} = -H - \lambda$ once more. Regard c as a curve in $\mathbb{R}^3$ and use the Frenet-equations (3.13) and equality for the first derivative of λ (B.6c) to get

$$\begin{aligned} & - \langle c, H \rangle T - \langle c, T \rangle H = \nabla_T \left(c^{\perp} \right) = -\nabla_T (H + \lambda) \\ & = T \left(|H|^2 + \langle H, \lambda \rangle \right) - \nu \left(\nabla_T |H| + \frac{1}{r} \nabla_T (r) |H| \right) - B \left(\tau (T) |H| - \frac{1}{r} \nabla_T (r) \langle B, \lambda \rangle \right), \end{aligned}$$

where T is the unit tangent vector field given by the orientation of c. Hence c is planar if and only if $r = const$, which means $e_1 \perp E$, or $\langle \lambda, B \rangle = 0$. But the latter case implies that $\lambda \in E$ and therefore, $e_1 \in E$.

The case $r = const$ was studied in Proposition 5.6. Let c be a solution to $c^{\perp} = -H - \lambda$ lying in a plane E with $e_1 \in E$. Obviously, the perturbed flow (5.1) is invariant under rotation ρ in $\mathbb{R}^3$ with $\rho(e_1) = e_1$[1]. Hence, without loss of generality E is the $y^1 y^2$-plane.

Angenent proved in [10] that there exists solutions $c \subset E$ to the perturbed flow (5.1). In fact, his results formulated in this setting are

Theorem 5.8. *There exists a solution $c \subset E$ to $c^{\perp} = -H - \lambda$ which is simple closed, symmetric with respect to reflection in the y^1-axis and stays away from the y^2-axis.*

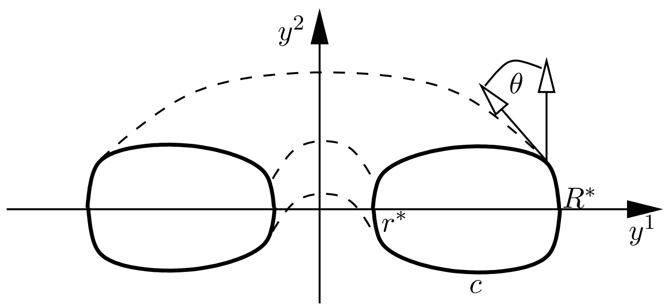

Figure 5.2.: Self-similar torus

The curve is convex as long as $r(x) > \sqrt{2}$ and $0 \leqslant \theta < \frac{\pi}{2}$, where θ is the angle described by the tangent vector and the y^2-axis (see Figure 5.2.), but it has not to be convex in general. So, the next question to ask is, if the generated surface satisfies some nice curvature conditions like *mean-convexity*[2].

[1] $\langle \rho(c), e_1 \rangle = \langle \rho(c) \rho(e_1) \rangle = \langle c, e_1 \rangle = r$ and $\rho\left(e_1^{\perp}\right) = \rho(e_1) - \langle e_1, T \rangle \rho(T) = e_1 - \langle e_1, T \rangle \rho(T)$ imply that $\rho(\lambda) = \lambda_\rho$, where λ_ρ is the perturbation vector for the transported curve $\rho(c)$.

[2] A hypersurface M^n is called *mean-convex*, if the sum of the two smallest principle curvatures is greater than zero: $\lambda_1 + \lambda_2 > 0$ if $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ (see for example [41]).

Angenent showed that the curve intersects the y^1-axis at some points R^* and r^*. It follows from his calculations that the scalar mean curvature of the generated surface has different signs in these intersection-points: Let $\bar{\nu}$ be the inwarding unit normal vector field of M and $k,\ l$ the principle curvatures, then $k+l>0$ on the orbit of radius R^*, $k+l<0$ on the orbit of radius r^*, i. e. M is not mean-convex.

After studying the planar self-shrinkers to the perturbed flow, the spherical self-shrinkers will be characterized in Corollary 5.10 and it will be shown that they are planar, too. The following theorem was proved by Smoczyk in [57, Theorem 1.1]:

Theorem 5.9. *Let $M^n \subset \mathbb{R}^m$, $n \geqslant 2$, be a compact self-shrinker. Then M is spherical if and only if $\mathcal{H} \neq 0$ and $\bar{\nabla}^{\perp}\bar{\nu} = 0$, where $\bar{\nu} := \frac{1}{|\mathcal{H}|}\mathcal{H}$ is the principle unit normal vector field of M.*

It implies:

Corollary 5.10. *$c(x) = (1, \cos x, \sin x)$, $x \in [0, 2\pi)$ is up to scaling and reparametrization the only spherical self-shrinker,i. e. the generated surface M is a flat torus.*

Proof. Let $c : S^1 \to S^2\left(\sqrt{m}\right)$ be any spherical self-shrinker on a sphere of radius $\sqrt{m}$. First note, that $\langle c, \lambda\rangle = -\frac{1}{r}\left\langle c, e_1^{\perp}\right\rangle = -\frac{1}{r}\langle c, e_1\rangle = -1$ leads to $m = 2$ since

$$0 = g^{ij}\nabla_i\nabla_j |c|^2 = 2\langle H, c\rangle + 2 = 2\left(-|c|^2 - \langle \lambda, c\rangle + 1\right) = 2(-m+2).$$

Secondly, one easily computes

$$|\lambda|^2 = \frac{1}{r^2}\left|e_1^{\perp}\right|^2 = \frac{1}{r^2}\left(1 - |\nabla r|^2\right), \quad |H|^2 = |c+\lambda|^2 = |\lambda|^2 = \frac{1}{r^2}\left(1 - |\nabla r|^2\right), \tag{5.9a}$$

$$|\mathcal{H}|^2 = |H+\lambda|^2 = |c|^2 = 2, \quad 2\langle H, \lambda\rangle = |\mathcal{H}|^2 - 2|H|^2 = 2\left(1 - \frac{1-|\nabla r|^2}{r^2}\right). \tag{5.9b}$$

Since the generated surfaces M_t evolving under the mean curvature flow are of codimension two, Theorem 5.9 leads to

$$\bar{\nabla}^{\perp}\bar{\nu} = 0 \Longleftrightarrow \bar{\tau} = 0, \tag{5.10}$$

where $\bar{\nu} = \frac{1}{|\mathcal{H}|}\mathcal{H} = \frac{1}{\sqrt{2}}(H+\lambda)$ is the principle unit normal vector field of M, $\bar{\tau} := \left\langle \bar{\nabla}^{\perp}\bar{\nu}, \bar{B}\right\rangle$ is the torsion 1-form, $\bar{B} := J\bar{\nu}$ is the unit binormal vector field (here, J denotes the complex structure on the normal bundle NM).

Now, it is necessary to compute the torsion of the surface in terms of the curve's torsion. Therefore, note that $\bar{\tau}\left(\frac{\partial}{\partial\phi}\right) = 0$, where ϕ is the angle-parameter of the rotation[3]. Hence, (5.10) implies

$$\bar{\tau}_1 = \bar{\tau}\left(\frac{\partial}{\partial x}\right) = \left\langle \bar{\nabla}_1\bar{\nu}, \bar{B}\right\rangle_{\mathbb{R}^4} = \frac{1}{2}\left\langle \frac{\partial}{\partial x}(H+\lambda), J_{Nc}(H+\lambda)\right\rangle_{\mathbb{R}^3},$$

[3] $F(x,\phi) = \left(r(x)\cos\phi, c^2(x), c^3(x), r(x)\sin\phi\right)$ in euclidean coordinates on $\mathbb{R}^4$, $F(x,\phi) = \left(r(x), c^2(x), c^3(x), \phi\right)$ in cylindrical coordinates on $\mathbb{R}^4$. Compare Section B.1.

where the normal space $N_{F(x^1,x^2)}M$ is identified with the normal space of the generating curve $N_{c(x^1)}c$ and J_{Nc} is the complex structure on the normal bundle of c.

As usual, B denotes the unit binormal of the curve. If $H \neq 0$, the Frenet equations (3.13) and the first (covariant) derivative of λ along the curve (cf. (B.6c)) yield

$$\begin{aligned} 2\bar{\tau}_1 &= \frac{\partial |H|}{\partial x} \langle \nu, J_{Nc}\lambda\rangle + |H|\,\tau_1 \langle B, J_{Nc}(H+\lambda)\rangle + \frac{1}{r}\frac{\partial r}{\partial x}\left(-\langle \lambda, J_{Nc}H\rangle + \langle H, J_{Nc}\lambda\rangle\right) \\ &= -\frac{\partial |H|}{\partial x^1}\langle B,\lambda\rangle + |H|^2\tau_1 + |H|\,\tau_1\langle \nu,\lambda\rangle - 2\frac{1}{r}\frac{\partial r}{\partial x^1}|H|\langle B,\lambda\rangle \\ &= \tau_1\left(|H|^2 + \langle H,\lambda\rangle\right) - \langle B,\lambda\rangle\left(2|H|\frac{1}{r}\frac{\partial r}{\partial x^1} + \frac{\partial |H|}{\partial x^1}\right). \end{aligned} \tag{5.11}$$

On the other hand, the torsion of a curve can be computed as the determinant of the first three partial derivatives[4] of the curve or equivalently (T denotes the unit tangent vector)

$$\begin{aligned} |H|^2\tau_1 &= \det(T, H, \nabla_1 H) = \det(T, c+\lambda, \nabla_1(c+\lambda)) = \det(T, c+\lambda, \nabla_1\lambda) \\ &= -\frac{1}{r}\frac{\partial r}{\partial x}\det(T, c, \lambda) = -\frac{1}{r}\frac{\partial r}{\partial x}\langle T\times c, \lambda\rangle = -\frac{1}{r}\frac{\partial r}{\partial x}|H|\langle B,\lambda\rangle \end{aligned} \tag{5.12}$$

Using $|H|^2 + \langle H,\lambda\rangle = 1$, we muliply (5.11) with $|H|^2$, and (5.10) yields

$$0 = -\langle B,\lambda\rangle\left(3|H|\frac{1}{r}\frac{\partial r}{\partial x} + \frac{\partial |H|}{\partial x}\right), \tag{5.13}$$

where the term in the bracket computes to

$$\begin{aligned} 3|H|\frac{1}{r}\frac{\partial r}{\partial x} + \frac{\partial |H|}{\partial x} &= 3\frac{\sqrt{1-|\nabla r|^2}}{r^2}\frac{\partial r}{\partial x} - \frac{\nabla_m r \nabla_1 \nabla^m r}{r\sqrt{1-|\nabla r|^2}} - \frac{\sqrt{1-|\nabla r|^2}}{r^2}\frac{\partial r}{\partial x} \\ &= 2\frac{\sqrt{1-|\nabla r|^2}}{r^2}\frac{\partial r}{\partial x} + \frac{\langle H,\lambda\rangle}{\sqrt{1-|\nabla r|^2}}\frac{\partial r}{\partial x} = \frac{\partial r}{\partial x}\left(\frac{\sqrt{1-|\nabla r|^2}}{r^2} + \frac{1}{\sqrt{1-|\nabla r|^2}}\right). \end{aligned} \tag{5.14}$$

The term $\frac{\sqrt{1-|\nabla r|^2}}{r^2} + \frac{1}{\sqrt{1-|\nabla r|^2}}$ is strictly positive whenever $H \neq 0$, hence, (5.13) implies $\langle B,\lambda\rangle = 0$ or $r = const$. The corollary follows after excluding the first case, because if r is constant, the curve $c = \left(r, c^2, c^3\right) =: (r,\gamma)$ is a solution to $c^\perp = -H - \lambda$ if and only if, $r = 1$ and γ is a spherical self-shrinker to the mean curvature flow, hence γ is the unit circle.

If $\langle B,\lambda\rangle = 0$, (5.12) implies that τ vanishes whenever $H \neq 0$. Since c is also a subset of $S^2\left(\sqrt{2}\right)$, the chain rule for the mean curvature vector implies that the curvature of c as a subset of $\mathbb{R}^3$ cannot vanish. Hence, Lemma 3.6 shows that c has to be totally

[4] $\tau_1 = \frac{1}{|H|^2|\dot{c}|^6}\det(\dot{c}, \ddot{c}, \dddot{c})$, $\dot{}$ denotes the partial derivative with respect to x.

planar, for instance, let E denote the plane with $c \subset E$. Then B is normal to E, and $\langle \lambda, B \rangle = 0$ yields that $\langle e_1, B \rangle = 0$, i. e. $e_1 \in E$. But $r > 0$ is a necessary condition on c, which contraditcs $c = E \cap S^2\left(\sqrt{2}\right)$. Hence the case $\langle \lambda, B \rangle = 0$ cannot occur for spherical self-shrinker. $\square$

A. Evolution Equations

In this chapter, the evolution equations used in Chapter 2 are derived in details. The equations that will be deduced are valid for every immersed starting manifold M_0 in euclidean space. For the sake of completeness we recall some notations.

A.1. Notations

The local coordinates on M are denoted by $\left(x^i\right)_{i=1,\dots,n}$, the coordinates on $\mathbb{R}^m$ by $(y^\alpha)_{\alpha=1,\dots,m}$. Doubled latin and greek indices are summed from 1 to n resp. from 1 to m. In local coordinates the *differential* dF is given by

$$dF = F_i^\alpha \frac{\partial}{\partial y^\alpha} \otimes dx^i,$$

where $F^\alpha := y^\alpha (F)$ and $F_i^\alpha := \frac{\partial F^\alpha}{\partial x^i}$.

Let $\delta = \delta_{\alpha\beta} dy^\alpha \otimes dy^\beta$ be the euclidean metric on $\mathbb{R}^m$ then the coefficients of the induced metric on M are

$$\bar{g}_{ij} = \delta_{\alpha\beta} F_i^\alpha F_j^\beta.$$

As usual, the Christoffel symbols are

$$\bar{\Gamma}_{ij}^k = \frac{1}{2} \bar{g}^{kl} \left(\bar{g}_{il,j} + \bar{g}_{jl,i} - \bar{g}_{ij,l}\right),$$

where $\bar{g}_{il,j} = \frac{\partial \bar{g}_{il}}{\partial x^j}$.

The Levi-Civita connection of M is denoted by $\bar{\nabla}$. The canonically induced connections on the pullback of the surrounding space $F^{-1}T\mathbb{R}^m$ or on the cotangent bundle T^*M of M are also denoted by $\bar{\nabla}$. $\bar{\Delta}$ is the Laplace-Beltrami-Operator. Then the *second fundamental form* is defined by

$$\mathcal{A} = \bar{\nabla} dF =: \mathcal{A}_{ij}^\alpha \frac{\partial}{\partial y^\alpha} \otimes dx^i \otimes dx^j \text{ with } \mathcal{A}_{ij}^\alpha = F_{ij}^\alpha - F_m^\alpha \bar{\Gamma}_{ij}^m,$$

where $F_{ij}^\alpha := \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j}$. By definition $\mathcal{A}$ is a section in $F^{-1}T\mathbb{R}^m \otimes T^*M \otimes T^*M$ and a standard computation shows that $\mathcal{A}$ is normal, i. e $\mathcal{A} \in \Gamma\left(NM \otimes T^*M \otimes T^*M\right)$, where NM denotes the normal bundle of M. In particular, the mean curvature vector $\mathcal{H} = \mathcal{H}^\alpha \frac{\partial}{\partial y^\alpha}$ with $\mathcal{H}^\alpha = \bar{g}^{ij} \mathcal{A}_{ij}^\alpha$ satisfies $\delta_{\alpha\beta} \mathcal{H}^\alpha F_j^\beta = 0$ for all j.

The Riemannian curvature tensor on M is given by

$$\bar{R}\left(W,Z,X,Y\right)=\bar{g}\left(\bar{\nabla}_X\bar{\nabla}_Y Z-\bar{\nabla}_Y\bar{\nabla}_X Z-\bar{\nabla}_{[X,Y]},W\right)$$

with local expression

$$\bar{R}_{klij}=\bar{R}_{klij}\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)=\bar{g}_{kn}\left(\frac{\partial\bar{\Gamma}^n_{jl}}{\partial x^i}-\frac{\partial\bar{\Gamma}^n_{il}}{\partial x^j}+\bar{\Gamma}^n_{im}\bar{\Gamma}^m_{jl}-\bar{\Gamma}^n_{jm}\bar{\Gamma}^m_{il}\right). \quad \text{(A.1)}$$

Since M is immersed in euclidean space, the Gauß equation on M simplify to

$$\bar{R}_{klij}=\delta_{\alpha\beta}\left(\mathcal{A}^\alpha_{ki}\mathcal{A}^\beta_{lj}-\mathcal{A}^\alpha_{kj}\mathcal{A}^\beta_{li}\right) \quad \text{(A.2)}$$

Furthermore, we have the Codazzi equation

$$\bar{\nabla}_i\mathcal{A}^\alpha_{jk}=\bar{\nabla}_j\mathcal{A}^\alpha_{ik}-F^\alpha_m\bar{R}^m{}_{kij}, \quad \text{(A.3)}$$

the rule for interchanging covariant derivatives

$$\bar{\nabla}_k\bar{\nabla}_l\mathcal{A}^\alpha_{ij}=\bar{\nabla}_l\bar{\nabla}_k\mathcal{A}^\alpha_{ij}-\mathcal{A}^\alpha_{im}\bar{R}^m{}_{jkl}-\mathcal{A}^\alpha_{mj}\bar{R}^m{}_{ikl} \quad \text{(A.4)}$$

and the Bianchi identities

$$\bar{R}_{klij}+\bar{R}_{kijl}+\bar{R}_{kjli}=0 \quad \text{(A.5a)}$$

$$\frac{\partial}{\partial x^m}\bar{R}_{klij}+\frac{\partial}{\partial x^l}\bar{R}_{mkij}+\frac{\partial}{\partial x^k}\bar{R}_{lmij}=0. \quad \text{(A.5b)}$$

The Codazzi equation, the rule for interganging derivatives and the second Bianchi identity the Simons' identity imply

$$\begin{aligned}\bar{\nabla}_l\bar{\nabla}_k\mathcal{H}^\alpha=&\bar{\Delta}\mathcal{A}^\alpha_{lk}-2\bar{g}^{ij}\mathcal{A}^\alpha_{jm}\bar{R}^m{}_{kli}-\mathcal{A}^\alpha_{mk}\bar{R}^m_l-\mathcal{A}^\alpha_{lm}\bar{R}^m_k\\&-F^\alpha_m\left(\bar{\nabla}_l\bar{R}^m_k+\bar{\nabla}_k\bar{R}^m_l-\bar{\nabla}^m\bar{R}_{kl}\right)\end{aligned} \quad \text{(A.6)}$$

A.2. Evolution equations for an immersed manifold in euclidean space

Since F_t is a smooth one-parameter family evolving under mean curvature flow, we get

$$\frac{\partial}{\partial t}F^\alpha_i=\nabla_i\frac{\partial}{\partial t}F^\alpha=\bar{\nabla}_i\mathcal{H}$$

and hence

$$\begin{aligned}\frac{\partial}{\partial t}\bar{g}_{ij}&=\delta_{\alpha\beta}\frac{\partial}{\partial t}F^\alpha_iF^\beta_j+\delta_{\alpha\beta}F^\alpha_i\frac{\partial}{\partial t}F^\beta_j=\delta_{\alpha\beta}\bar{\nabla}_i\mathcal{H}^\alpha F^\beta_j+\delta_{\alpha\beta}\bar{\nabla}_j\mathcal{H}^\beta F^\alpha_i\\&=-2\delta_{\alpha\beta}\mathcal{H}^\alpha\mathcal{A}^\beta_{ij}=:-2\bar{a}_{ij}\end{aligned} \quad \text{(A.7)}$$

$$\frac{\partial}{\partial t}\mathrm{d}\eta=\frac{1}{2\sqrt{\det g}}\bar{g}^{ij}\frac{\partial}{\partial t}\bar{g}_{ij}\det g\,\mathrm{dx}=\frac{1}{2}\left(-2\bar{g}^{ij}\bar{a}_{ij}\right)\mathrm{d}\eta=-\left|\mathcal{H}\right|^2\mathrm{d}\eta, \quad \text{(A.8)}$$

where $d\eta = \sqrt{\det \bar{g}}\, dx$ is the volume form of M. The Christoffel symbols fulfil

$$\frac{\partial}{\partial t}\bar{\Gamma}^k_{ij} = \frac{1}{2}\bar{g}^{kl}\left(\bar{\nabla}_i \frac{\partial}{\partial t}\bar{g}_{lj} + \bar{\nabla}_j \frac{\partial}{\partial t}\bar{g}_{li} - \bar{\nabla}_l \frac{\partial}{\partial t}\bar{g}_{ij}\right) = -\bar{g}^{kl}\left(\bar{\nabla}_i \bar{a}_{lj} + \bar{\nabla}_j \bar{a}_{il} - \bar{\nabla}_l \bar{a}_{ij}\right). \quad \text{(A.9)}$$

This is true in normal coordinates and since (A.9) is a equation between tensors, it is independent of the choice of the local coordinates.

Now, the evolution equations of the second fundamental form and its higher covariant derivatives can be computed. The expression of the Riemannian curvature tensor in terms of the Christoffel symbols (A.1), equation (A.9) and Simons' equation (A.6) imply

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}^\alpha_{lk} =& \bar{\nabla}_l \frac{\partial}{\partial t} F^\alpha_k - F^\alpha_m \bar{R}^m{}_{ktl} = \bar{\nabla}_l \bar{\nabla}_k \mathcal{H}^\alpha - F^\alpha_m \frac{\partial}{\partial t}\bar{\Gamma}^m_{lk} \\ =& \bar{\Delta}\mathcal{A}^\alpha_{lk} - 2\bar{g}^{ij}\mathcal{A}^\alpha_{jm}\bar{R}^m{}_{kli} - \mathcal{A}^\alpha_{mk}\bar{R}^m_l - \mathcal{A}^\alpha_{lm}\bar{R}^m_k \\ & - F^\alpha_m\left(\bar{\nabla}_l \bar{R}^m_k + \bar{\nabla}_k \bar{R}^m_l - \bar{\nabla}^m \bar{R}_{kl} + \bar{\nabla}_l \bar{a}^m_k + \bar{\nabla}_k \bar{a}^m_l - \bar{\nabla}_n \bar{a}_{lk}\right),\end{aligned}$$

where $\bar{a}_{ij} = \delta_{\alpha\beta}\mathcal{A}^\alpha_{ij}\mathcal{H}^\beta$ and $\bar{b}_{ij} := \delta_{\alpha\beta}\bar{g}^{kl}\mathcal{A}^\alpha_{ik}\mathcal{A}^\beta_{jl}$. The Gauß equation (A.2) gives $\bar{R}_{ij} = \bar{a}_{ij} - \bar{b}_{ij}$, and the evolution equation of $\mathcal{A}^\alpha_{kl}$ becomes:

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}^\alpha_{kl} =& \bar{\Delta}\mathcal{A}^\alpha_{lk} - 2\bar{g}^{ij}\mathcal{A}^\alpha_{jm}\bar{R}^m{}_{kli} - \mathcal{A}^\alpha_{mk}\bar{R}^m_l - \mathcal{A}^\alpha_{lm}\bar{R}^m_k \\ & + F^\alpha_m\left(\bar{\nabla}_l \bar{b}^m_k + \bar{\nabla}_k \bar{b}^m_l - \bar{\nabla}^m \bar{b}_{kl}\right).\end{aligned} \quad \text{(A.10)}$$

From the evolution equation for the metric (A.7), it follows that

$$\frac{\partial}{\partial t}|\mathcal{A}|^2 = \bar{\Delta}|\mathcal{A}|^2 - 2\left|\bar{\nabla}\mathcal{A}\right|^2 - 4\bar{g}^{ij}\mathcal{A}_{\alpha jm}\bar{R}^m{}_{kli}\mathcal{A}^{\alpha kl} - 4\mathcal{A}_{\alpha lm}\mathcal{A}^{\alpha lk}\bar{R}^m_k + 4\bar{a}^{ij}\bar{b}_{ij}.$$

Furthermore, (A.10) and (A.7) imply

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{H}^\alpha =& \Delta\mathcal{H}^\alpha + F^\alpha_m\left(2\nabla_n \bar{b}^{mn} - \nabla^m|\mathcal{A}|^2\right) + 2\bar{a}^{kl}\mathcal{A}^\alpha_{kl} \\ =& \Delta\mathcal{H}^\alpha + 2F^\alpha_m\mathcal{A}^{\beta im}\nabla_i\mathcal{H}_\beta + 2\bar{a}^{kl}\mathcal{A}^\alpha_{kl}\end{aligned} \quad \text{(A.11)}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}|\mathcal{H}|^2 &= \Delta|\mathcal{H}|^2 - 2|\nabla\mathcal{H}|^2 + 4\bar{a}^{kl}\bar{a}_{kl} \\ &= \Delta|\mathcal{H}|^2 - 2|\nabla_i H^\alpha + \bar{a}^m_i F^\alpha_m|^2 + 2\bar{a}^{ij}\bar{a}_{ij}.\end{aligned} \quad \text{(A.12)}$$

To give an idea, how complicated the evolution equations for the higher derivatives of the second fundamental form become even in the euclidean space, the evolution equation for the first derivative of the second fundamental form is determined, too. Using Codazzi's equation (A.3), one gets

$$\begin{aligned}\bar{\nabla}_l \bar{b}^m_k + \bar{\nabla}_k \bar{b}^m_l - \bar{\nabla}^m \bar{b}_{kl} &= \bar{\nabla}_l\left(\mathcal{A}_{\alpha ik}\mathcal{A}^{\alpha im}\right) + \bar{\nabla}_k\left(\mathcal{A}_{\alpha il}\mathcal{A}^{\alpha im}\right) - \bar{\nabla}^m\left(\mathcal{A}^{\alpha i}_l\mathcal{A}_{\alpha ik}\right) \\ &= 2\mathcal{A}^{\alpha im}\bar{\nabla}_i\mathcal{A}_{\alpha kl}\end{aligned}$$

and (A.10) gives

$$\begin{aligned}\frac{\partial}{\partial t}\bar\nabla_i\mathcal{A}^\alpha_{kl} =&\bar\nabla_i\frac{\partial}{\partial t}\mathcal{A}^\alpha_{kl}-\mathcal{A}^\alpha_{km}\frac{\partial}{\partial t}\bar\Gamma^m_{li}-\mathcal{A}^\alpha_{lm}\frac{\partial}{\partial t}\bar\Gamma^m_{ki}\\ =&\bar\nabla_i\left(\bar\Delta\mathcal{A}^\alpha_{lk}-2\bar g^{rs}\mathcal{A}^\alpha_{sm}\bar R^m{}_{klr}-\mathcal{A}^\alpha_{mk}\bar R^m_l-\mathcal{A}^\alpha_{lm}\bar R^m_k\right)\\ &+2\mathcal{A}^\alpha_{im}\mathcal{A}^{\beta nm}\bar\nabla_n\mathcal{A}_{\beta kl}+2F^\alpha_m\bar\nabla_i\mathcal{A}^{\beta nm}\bar\nabla_n\mathcal{A}_{\beta kl}+2F^\alpha_m\mathcal{A}^{\beta nm}\bar\nabla_i\bar\nabla_n\mathcal{A}_{\beta kl}\\ &-\mathcal{A}^\alpha_{km}\frac{\partial}{\partial t}\bar\Gamma^m_{li}-\mathcal{A}^\alpha_{lm}\frac{\partial}{\partial t}\bar\Gamma^m_{ki}\end{aligned}$$

Interchanging the covariant derivative and the Laplace operator leads to

$$\begin{aligned}\bar\nabla_i\bar\Delta\mathcal{A}^\alpha_{kl} =&\bar g^{mn}\bar\nabla_m\bar\nabla_i\bar\nabla_n\mathcal{A}^\alpha_{kl}-\bar\nabla_s\mathcal{A}^\alpha_{kl}\bar R^s_i+\bar\nabla_m\mathcal{A}^\alpha_{ks}\bar R^s{}_l{}^m{}_i+\bar\nabla_m\mathcal{A}^\alpha_{ls}\bar R^s{}_k{}^m{}_i\\ =&\bar\Delta\bar\nabla_i\mathcal{A}^\alpha_{kl}+\bar g^{mn}\bar\nabla_m\left(\mathcal{A}^\alpha_{ks}\bar R^s{}_{lni}+\mathcal{A}^\alpha_{sl}\bar R^s{}_{kni}\right)\\ &-\bar\nabla_s\mathcal{A}^\alpha_{kl}\bar R^s_i+\bar\nabla_m\mathcal{A}^\alpha_{ks}\bar R^s{}_l{}^m{}_i+\bar\nabla_m\mathcal{A}^\alpha_{ls}\bar R^s{}_k{}^m{}_i\\ =&\bar\Delta\bar\nabla_i\mathcal{A}^\alpha_{kl}+2\bar\nabla_m\mathcal{A}^\alpha_{ks}\bar R^s{}_l{}^m{}_i+2\bar\nabla_m\mathcal{A}^\alpha_{ls}\bar R^s{}_k{}^m{}_i-\bar\nabla_s\mathcal{A}^\alpha_{kl}\bar R^s_i\\ &+\mathcal{A}^\alpha_{ks}\bar\nabla_m\bar R^s{}_l{}^m{}_i+\mathcal{A}^\alpha_{ls}\bar\nabla_m\bar R^s{}_k{}^m{}_i.\end{aligned}$$

Inserting the evolution equation for the Christoffel symbols (A.9) yields

$$\begin{aligned}\frac{\partial}{\partial t}\bar\nabla_i\mathcal{A}^\alpha_{kl} =&\bar\Delta\bar\nabla_i\mathcal{A}^\alpha_{kl}+2\bar\nabla_m\mathcal{A}^\alpha_{ks}\bar R^s{}_l{}^m{}_i+2\bar\nabla_m\mathcal{A}^\alpha_{ls}\bar R^s{}_k{}^m{}_i-\bar\nabla_s\mathcal{A}^\alpha_{kl}\bar R^s_i\\ &+\mathcal{A}^\alpha_{ks}\bar\nabla_m\bar R^s{}_l{}^m{}_i+\mathcal{A}^\alpha_{ls}\bar\nabla_m\bar R^s{}_k{}^m{}_i-\bar\nabla_i\left(2\bar g^{rs}\mathcal{A}^\alpha_{sm}\bar R^m{}_{klr}+\mathcal{A}^\alpha_{mk}\bar R^m_l+\mathcal{A}^\alpha_{lm}\bar R^m_k\right)\\ &+2\mathcal{A}^\alpha_{im}\mathcal{A}^{\beta nm}\bar\nabla_n\mathcal{A}_{\beta kl}+2F^\alpha_m\bar\nabla_i\mathcal{A}^{\beta nm}\bar\nabla_n\mathcal{A}_{\beta kl}+2F^\alpha_m\mathcal{A}^{\beta nm}\bar\nabla_i\bar\nabla_n\mathcal{A}_{\beta kl}\\ &-\mathcal{A}^\alpha_{km}\frac{\partial}{\partial t}\bar\Gamma^m_{li}-\mathcal{A}^\alpha_{lm}\frac{\partial}{\partial t}\bar\Gamma^m_{ki}\\ =&\bar\Delta\bar\nabla_i\mathcal{A}^\alpha_{kl}+2\bar\nabla_m\mathcal{A}^\alpha_{ks}\bar R^s{}_l{}^m{}_i+2\bar\nabla_m\mathcal{A}^\alpha_{ls}\bar R^s{}_k{}^m{}_i-\bar\nabla_s\mathcal{A}^\alpha_{kl}\bar R^s_i\\ &+\mathcal{A}^\alpha_{ks}\left(\bar\nabla^s\bar R_{li}-\bar\nabla_l\bar R^s_i-\bar\nabla_i\bar R^s_l\right)+\mathcal{A}^\alpha_{ls}\left(\bar\nabla^s\bar R_{ki}-\bar\nabla_k\bar R^s_i-\bar\nabla_i\bar R^s_k\right)\\ &+2\bar\nabla_i\mathcal{A}^\alpha_{sm}\bar R^m{}_k{}^s{}_l-\bar\nabla_i\mathcal{A}^\alpha_{mk}\bar R^m_l-\bar\nabla_i\mathcal{A}^\alpha_{ml}\bar R^m_k+2\mathcal{A}^\alpha_{sm}\bar\nabla_i\bar R^m{}_k{}^s{}_l\\ &+2\mathcal{A}^\alpha_{im}\mathcal{A}^{\beta nm}\bar\nabla_n\mathcal{A}_{\beta kl}+2F^\alpha_m\bar\nabla_i\mathcal{A}^{\beta nm}\bar\nabla_n\mathcal{A}_{\beta kl}+2F^\alpha_m\mathcal{A}^{\beta nm}\bar\nabla_i\bar\nabla_n\mathcal{A}_{\beta kl}\\ &+\mathcal{A}^\alpha_{ks}\left(\bar\nabla_i\bar a^s_l+\bar\nabla_l\bar a^s_i-\bar\nabla^s\bar a_{li}\right)+\mathcal{A}^\alpha_{ls}\left(\bar\nabla_i\bar a^s_k+\bar\nabla_k\bar a^s_i-\bar\nabla^s\bar a_{ki}\right)\\ =&\bar\Delta\bar\nabla_i\mathcal{A}^\alpha_{kl}+2\bar\nabla_m\mathcal{A}^\alpha_{ks}\bar R^s{}_l{}^m{}_i+2\bar\nabla_m\mathcal{A}^\alpha_{ls}\bar R^s{}_k{}^m{}_i-\bar\nabla_s\mathcal{A}^\alpha_{kl}\bar R^s_i+2\mathcal{A}^\alpha_{ks}\mathcal{A}^{\beta sm}\bar\nabla_m\mathcal{A}_{\beta li}\\ &+2\mathcal{A}^\alpha_{ls}\mathcal{A}^{\beta sm}\bar\nabla_m\mathcal{A}_{\beta ki}+2\bar\nabla_i\mathcal{A}^\alpha_{sm}\bar R^m{}_k{}^s{}_l-\bar\nabla_i\mathcal{A}^\alpha_{mk}\bar R^m_l-\bar\nabla_i\mathcal{A}^\alpha_{ml}\bar R^m_k\\ &+2\mathcal{A}^\alpha_{sm}\bar\nabla_i\bar R^m{}_k{}^s{}_l+2\mathcal{A}^\alpha_{im}\mathcal{A}^{\beta nm}\bar\nabla_n\mathcal{A}_{\beta kl}\\ &+2F^\alpha_m\bar\nabla_i\mathcal{A}^{\beta nm}\bar\nabla_n\mathcal{A}_{\beta kl}+2F^\alpha_m\mathcal{A}^{\beta nm}\bar\nabla_i\bar\nabla_n\mathcal{A}_{\beta kl},\end{aligned}$$

where again $\bar R_{li}=\bar a_{li}-\bar b_{li}$ and $\bar\nabla_l\bar b^m_k+\bar\nabla_k\bar b^m_l-\bar\nabla^m\bar b_{kl}=2\mathcal{A}^{\alpha im}\bar\nabla_i\mathcal{A}_{\alpha kl}$ was used.

Simplifying these terms gives

$$\begin{aligned}\frac{\partial}{\partial t}\bar{\nabla}_i\mathcal{A}^\alpha_{kl} =& \bar{\Delta}\bar{\nabla}_i\mathcal{A}^\alpha_{kl} + 2\bar{\nabla}_m\mathcal{A}^\alpha_{ks}\bar{R}^{s\ m}_{\ l\ \ i} + 2\bar{\nabla}_m\mathcal{A}^\alpha_{ls}\bar{R}^{s\ m}_{\ k\ \ i} - \bar{\nabla}_s\mathcal{A}^\alpha_{kl}\bar{R}^s_i \\ &+ 2\bar{\nabla}_i\mathcal{A}^\alpha_{sm}\bar{R}^{m\ s}_{\ k\ \ l} - \bar{\nabla}_i\mathcal{A}^\alpha_{mk}\bar{R}^m_l - \bar{\nabla}_i\mathcal{A}^\alpha_{ml}\bar{R}^m_k \\ &+ 2\mathcal{A}^\alpha_{ks}\mathcal{A}^{\beta sm}\bar{\nabla}_i\mathcal{A}_{\beta lm} + 2\mathcal{A}^\alpha_{ls}\mathcal{A}^{\beta sm}\bar{\nabla}_i\mathcal{A}_{\beta km} \\ &+ 2\mathcal{A}^\alpha_{im}\mathcal{A}^{\beta nm}\bar{\nabla}_n\mathcal{A}_{\beta kl} + 2F^\alpha_m\bar{\nabla}_i\mathcal{A}^{\beta nm}\bar{\nabla}_n\mathcal{A}_{\beta kl} + 2F^\alpha_m\mathcal{A}^{\beta nm}\bar{\nabla}_i\bar{\nabla}_n\mathcal{A}_{\beta kl} \\ &+ 2\mathcal{A}^\alpha_{sm}\bar{\nabla}_i\bar{R}^{m\ s}_{\ k\ \ l}. \end{aligned} \tag{A.13}$$

The norm evolves like

$$\begin{aligned}\frac{\partial}{\partial t}\left|\bar{\nabla}_i\mathcal{A}^\alpha_{kl}\right|^2 =& \bar{\Delta}\left|\bar{\nabla}_i\mathcal{A}^\alpha_{kl}\right| - 2\left|\bar{\nabla}_j\bar{\nabla}_i\mathcal{A}^\alpha_{kl}\right|^2 + 8\bar{\nabla}^i\mathcal{A}^{\alpha kl}\bar{\nabla}^j\mathcal{A}_{\alpha ks}\bar{R}^s{}_{lji} - 2\bar{R}^{ij}\bar{\nabla}_i\mathcal{A}^{\alpha kl}\bar{\nabla}_j\mathcal{A}_{\alpha kl} \\ &+ 4\bar{\nabla}^i\mathcal{A}^{\alpha kl}\bar{\nabla}_i\mathcal{A}^{sm}_\alpha\bar{R}_{mksl} - 4\bar{\nabla}^i\mathcal{A}^{\alpha kl}\bar{\nabla}_i\mathcal{A}_{\alpha mk}\bar{R}^m_l \\ &+ 8\bar{\nabla}^i\mathcal{A}^{\alpha kl}2\mathcal{A}_{\alpha ks}\mathcal{A}^{\beta sm}\bar{\nabla}_i\mathcal{A}_{\beta lm} + 4\bar{\nabla}^i\mathcal{A}^{\alpha kl}\mathcal{A}_{\alpha im}\mathcal{A}^{\beta nm}\bar{\nabla}_n\mathcal{A}_{\beta kl} \\ &+ 4\bar{\nabla}^i\mathcal{A}^{\alpha kl}\mathcal{A}^{sm}_\alpha\bar{\nabla}_i\bar{R}_{mksl} + 2\bar{a}^{ij}\bar{\nabla}_i\mathcal{A}^{\alpha kl}\bar{\nabla}_j\mathcal{A}_{\alpha kl} + 4\bar{a}^n_m\bar{\nabla}^i\mathcal{A}^{\alpha km}\bar{\nabla}_i\mathcal{A}_{\alpha kn} \\ &+ 4F^\alpha_m\bar{\nabla}_i\mathcal{A}^{\beta nm}\bar{\nabla}_n\mathcal{A}_{\beta kl}\bar{\nabla}^i\mathcal{A}^{kl}_\alpha + 4F^\alpha_m\mathcal{A}^{\beta nm}\bar{\nabla}_i\bar{\nabla}_n\mathcal{A}_{\beta kl}\bar{\nabla}^i\mathcal{A}^{kl}_\alpha \\ =& \bar{\Delta}\left|\bar{\nabla}_i\mathcal{A}^\alpha_{kl}\right| - 2\left|\bar{\nabla}_j\bar{\nabla}_i\mathcal{A}^\alpha_{kl}\right|^2 \\ &+ 8\left|\bar{\nabla}_n\mathcal{A}_{\alpha km}\mathcal{A}^{\beta mn}\right|^2 - 8\bar{\nabla}^i\mathcal{A}^{\alpha kl}\bar{\nabla}^j\mathcal{A}_{\alpha ks}\mathcal{A}^{\beta s}_i\mathcal{A}_{\beta lj} + 2b^{ij}\bar{\nabla}_i\mathcal{A}^{\alpha kl}\bar{\nabla}_j\mathcal{A}_{\alpha kl} \\ &+ 4\left|\bar{\nabla}_i\mathcal{A}^\alpha_{nm}\mathcal{A}^{\beta mn}\right|^2 - 4\bar{\nabla}^i\mathcal{A}^{\alpha kl}\bar{\nabla}_i\mathcal{A}^{sm}_\alpha\mathcal{A}^\beta_{ks}\mathcal{A}_{\beta lm} + 4b^m_n\bar{\nabla}^i\mathcal{A}^{\alpha kn}\bar{\nabla}_i\mathcal{A}_{\alpha mk} \\ &+ 8\left|\bar{\nabla}_i\mathcal{A}^{\alpha mk}\mathcal{A}_{\alpha ml}\right|^2 + 4\left|\bar{\nabla}^n\mathcal{A}^{\alpha kl}\mathcal{A}_{\alpha ni}\right|^2 + 4\bar{\nabla}^i\mathcal{A}^{\alpha kl}\mathcal{A}^{sm}_\alpha\bar{\nabla}_i\bar{R}_{mksl} \\ &- 4\mathcal{A}^{\alpha i}_m\mathcal{A}^{kl}_\alpha\bar{\nabla}_i\mathcal{A}^{\beta nm}\bar{\nabla}_n\mathcal{A}_{\beta kl} + 4F^\alpha_m\bar{\nabla}^i\mathcal{A}^{kl}_\alpha\mathcal{A}^{\beta nm}\bar{\nabla}_i\bar{\nabla}_n\mathcal{A}_{\beta kl}\end{aligned} \tag{A.14}$$

More imprecisely: There exist constants d_1, d_2 only depending on the dimension and the codimension of M, such that

$$\frac{\partial}{\partial t}\left|\bar{\nabla}\mathcal{A}\right|^2 \leqslant \bar{\Delta}\left|\bar{\nabla}\mathcal{A}\right|^2 - 2\left|\bar{\nabla}^2\mathcal{A}\right|^2 + d_1\left|\mathcal{A}\right|^2\left|\bar{\nabla}\mathcal{A}\right|^2 + d_2\left|\mathcal{A}\right|\left|\bar{\nabla}\mathcal{A}\right|\left|\bar{\nabla}^2\mathcal{A}\right|. \tag{A.15}$$

B. Properties of rotationally symmetric surfaces

First, the perturbed flow for the generating manifolds c is determined via computing the second fundamental form of M in cylindrical coordinates. Secondly, the evolution equations stated in Chapter 3 are proved.

B.1. Cylindrical coordinates and the mean curvature vector

Let $c : [0, 2\pi)^n \to \mathbb{R}^m_{>0}$ be a smooth immersed compact manifold in the halfspace $\mathbb{R}^m_{>0} := \left\{y \in \mathbb{R}^m ; y^1 > 0\right\} \subset \mathbb{R}^{m+1}$. Define the immersed manifold $F : [0, 2\pi)^{n+1} \to M \subset \mathbb{R}^{m+1}$ generated by c such that M is invariant under all rotations in the $y^1 y^{m+1}$-plane:

$$
\begin{aligned}
M &:= F\left([0, 2\pi)^n \times [0, 2\pi[\right) \\
&:= \Big\{\left(c^1(x) \cos x^{n+1}, c^2(x), \ldots, c^{n+2}(x),\ c^1(x) \sin x^{n+1}\right) \in \mathbb{R}^{m+1}; \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in [0, 2\pi)^n, x^{n+1} \in [0, 2\pi[\Big\}.
\end{aligned}
$$

Let y^α, $\alpha = 1, \ldots, m+1$, be the euclidean and $\left(z^1, \ldots, z^{m+1}\right) = \left(r, y^2, \ldots, y^m, \phi\right)$ be the cylindrical coordinates on $\mathbb{R}^{m+1}$. Then $y^1 = r \cos\phi$, $y^{m+1} = r \sin\phi$ and $\frac{\partial}{\partial r} = \cos\phi \frac{\partial}{\partial y^1} + \sin\phi \frac{\partial}{\partial y^{m+1}}$. The coefficients of the euclidean metric $h_{\alpha\beta}$ and the Christoffel symbols $D^\alpha_{\beta\gamma}$ on $\mathbb{R}^{m+1}$ are given in cylindrical coordinates by

$$
\begin{aligned}
&h_{\alpha\beta}(z) = \delta_{\alpha\beta} \text{ for } \alpha, \beta = 1, \ldots, m, \quad h_{m+1\,m+1}(z) = \left(z^1\right)^2 \\
&D^1_{\alpha\beta}(z) = 0 \text{ for } \alpha, \beta = 1, \ldots, m, \quad D^1_{m+1\,m+1} = -z^1 \\
&D^{m+1}_{\alpha\beta}(z) = 0 \text{ for } \alpha, \beta = 1, \ldots, m+1, (m+1, 1) \neq (\alpha, \beta) \neq (1, m+1) \\
&D^{m+1}_{1\,m+1}(z) = \frac{1}{z^1} = D^{m+1}_{m+1\,1}(z) \\
&D^\gamma_{\alpha\beta} = 0 \text{ for } \gamma = 2, \ldots, m, \alpha, \beta = 1, \ldots, m+1
\end{aligned}
$$

Consider $\mathbb{R}^m_{>0}$ as a subspace of $\mathbb{R}^{m+1}$. The local coordinates on c are denoted by $x = \left(x^i\right)_{i=1,\ldots,n}$ (resp. on $\mathbb{R}^m$ by $\left(y^\alpha\right)_{\alpha=1,\ldots,m}$).

In cylindrical coordinates, F is written as $F\left(x, x^{n+1}\right) = \left(c^1(x), \ldots, c^m(x), x^{n+1}\right)$

and hence,

$$F_l^\alpha \frac{\partial}{\partial z^\alpha} := \frac{\partial F^\alpha}{\partial x^l}\frac{\partial}{\partial z^\alpha} = \left(\frac{\partial c^1}{\partial x^l}, \dots \frac{\partial c^m}{\partial x^l}, 0\right), \text{ for } l = 1, \dots n,$$
$$F_{n+1}^\alpha \frac{\partial}{\partial z^\alpha} = \frac{\partial F^\alpha}{\partial x^{n+1}}\frac{\partial}{\partial z^\alpha} = (0, \dots, 0, 1).$$

Thus, the induced metric $\bar{g}_{ij} = h_{\alpha\beta} F_i^\alpha F_j^\beta$ on M is given by

$$\bar{g}_{ij} = \delta_{\alpha\beta} c_i^\alpha c_j^\beta, \text{ for } i, j = 1, \dots, n, \quad \bar{g}_{i\,n+1} = 0, \quad \bar{g}_{n+1\,n+1} = r^2 := \left(c^1\right)^2,$$

and the coefficients of the induced metric on c are $g_{ij} = \delta_{\alpha\beta} c_i^\alpha c_j^\beta = \bar{g}_{ij}$ for $i, j = 1, \dots n$.

In the following, Γ_{ij}^k denote the Christoffel symbols on c. The Christoffel symbols $\bar{\Gamma}_{ij}^k$ on M are given by

$$\begin{aligned}
&\bar{\Gamma}_{ij}^k \left(x, x^{n+1}\right) = \Gamma_{ij}^k (x), \quad \bar{\Gamma}_{i\,n+1}^k \left(x, x^{n+1}\right) = 0 \quad \text{ for } k, i, j = 1, \dots, n, \\
&\bar{\Gamma}_{n+1\,n+1}^k = -g^{kl}(x)\left(c_l^1(x)\, c^1(x)\right) = -g^{kl} r(x) \frac{\partial r}{\partial x^l}(x) \text{ for } k = 1, \dots, n, \\
&\bar{\Gamma}_{ij}^{n+1}\left(x, x^{n+1}\right) = 0 \text{ for } i, j = 1, \dots, n, \quad \bar{\Gamma}_{n+1\,n+1}^{n+1}\left(x, x^{n+1}\right) = 0, \\
&\bar{\Gamma}_{i\,n+1}^{n+1}\left(x, x^{n+1}\right) = \frac{1}{\left(c^1(x)\right)^2} c_i^1(x)\, c^1(x) = \frac{1}{r(x)} \frac{\partial r}{\partial x^i}(x) \text{ for } i = 1, \dots, n.
\end{aligned}$$

The second fundamental form $\mathcal{A} = \sum_{\substack{1 \leqslant \alpha \leqslant m+1 \\ 1 \leqslant i,j \leqslant n+1}} \mathcal{A}_{ij}^\alpha \frac{\partial}{\partial y^\alpha} \otimes dx^i \otimes dx^j$ of M computes in cylindrical coordinates to

$$\begin{aligned}
\mathcal{A}_{ij}^\alpha \left(x, x^{n+1}\right) &= F_{ij}^\alpha - F_m^\alpha \bar{\Gamma}_{ij}^m + F_i^\gamma F_j^\beta D_{\gamma\beta}^\alpha = c_{ij}^\alpha - c_m^\alpha \Gamma_{ij}^m \\
&= A_{ij}^\alpha (x) \text{ for } \alpha = 1, \dots, m, \ i, j = 1, \dots, n \\
\mathcal{A}_{n+1\,n+1}^\alpha \left(x, x^{n+1}\right) &= -F_m^\alpha \bar{\Gamma}_{n+1 n+1}^m + F_{n+1}^\gamma F_{n+1}^\beta D_{\beta\gamma}^\alpha \\
&= c_m^\alpha g^{ml} c_l^1 c^1 - c^1 \delta^{1\alpha} = -c^1 \left(\delta^{1\alpha} - c_m^\alpha g^{ml} c_l^1\right) \text{ for } \alpha = 1, \dots, m \\
\mathcal{A}_{i\,n+1}^\alpha &= -F_m^\alpha \bar{\Gamma}_{i\,n+1}^m + F_i^\gamma F_{n+1}^\beta D_{\beta\gamma}^\alpha = 0 \text{ for } \alpha = 1, \dots, m, \ i = 1, \dots n \\
\mathcal{A}_{ij}^{m+1}\left(x, x^{n+1}\right) &= F_{ij}^{m+1} - F_m^{m+1} \bar{\Gamma}_{ij}^m + F_i^\gamma F_j^\beta D_{\gamma\beta}^{m+1} \\
&= -F_{n+1}^{m+1} \bar{\Gamma}_{ij}^{n+1} + F_i^1 F_j^{m+1} D_{1m+1}^{m+1} + F_i^{m+1} F_j^1 D_{m+11}^{m+1} \\
&= 0 \text{ for all } i, j = 1, \dots, n+1,
\end{aligned}$$

where $A = \sum_{\substack{1 \leqslant \alpha \leqslant m \\ 1 \leqslant i,j \leqslant n}} A_{ij}^\alpha \frac{\partial}{\partial y^\alpha} \otimes dx^i \otimes dx^j$ is the second fundamental form on c.

The mean curvature vector $\mathcal{H}$ of M is given by

$$\mathcal{H}^\alpha = \bar{g}^{ij} \mathcal{A}_{ij}^\alpha = g^{ij} A_{ij}^\alpha + \frac{1}{\left(c^1\right)^2} \mathcal{A}_{n+1\,n+1}^\alpha =: H^\alpha + \lambda^\alpha \text{ for } \alpha = 1, \dots, m, \tag{B.1}$$

$$\mathcal{H}^{m+1} = 0, \tag{B.2}$$

where H^α is the mean curvature vector of C, and $\lambda^\alpha := \frac{1}{(c^1)^2}\mathcal{A}^\alpha_{n+1\,n+1}$ denotes a vector due to the rotation:

$$\lambda^\alpha = -\frac{1}{c^1}\left(\delta^{1\alpha} - c^\alpha_m g^{ml} c^1_l\right) = -\frac{1}{c^1}\left(\frac{\partial}{\partial z^1} - \delta_{1\beta} c^\beta_m g^{ml} c^\alpha_l\right) = -\frac{1}{c^1}\left(\frac{\partial}{\partial z^1}\right)^\perp .$$

Here, $^\perp$ is the orthogonal projection to NM. From now on write

$$e_1 := \frac{\partial}{\partial z^1} = \cos z^{m+1}\frac{\partial}{\partial y^1} + \sin z^{m+1}\frac{\partial}{\partial y^{m+1}} \quad r = c^1(x) = \langle c(x), e_1\rangle .$$

Then the perturbation vector is given by

$$\lambda = -\frac{1}{r}\left(e_r\right)^\perp . \tag{B.3}$$

Note that here $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathbb{R}^{m+1}$, but $\langle\cdot,\cdot\rangle$ is used also for the scalar product given by the metric g on Tc or by the induced metric on $c^{-1}T\mathbb{R}^m$. It will become clear out of the context which scalar product is used.

Let $\bar\nabla$ (resp. ∇) denote the induced connection on $F^{-1}T\mathbb{R}^{m+1}$ (resp. $c^{-1}T\mathbb{R}^m$). Then the norm of the higher covariant derivatives of the second fundamental forms of c is bounded by the corresponding norm on M:

$$\left|\nabla^N A\right|^2 \leqslant \left|\bar\nabla^N \mathcal{A}\right|^2, \text{ for } N \leqslant 0. \tag{B.4}$$

This follows easily from the representation in cylindrical coordinates, here for $N = 1$:

$$\begin{aligned}
&h_{\alpha\beta}\bar g^{ij}\bar g^{rs}\bar g^{kl}\bar\nabla_k\mathcal{A}^\alpha_{ir}\bar\nabla_l\mathcal{A}^\alpha_{js}\\
&\quad=\delta_{\alpha\beta}g^{ij}g^{rs}g^{kl}\nabla_k A^\alpha_{ir}\nabla_l A^\alpha_{js} + r^2\bar g^{ij}\bar g^{rs}\bar g^{kl}\bar\nabla_k\mathcal{A}^{m+1}_{ir}\bar\nabla_l\mathcal{A}^{m+1}_{js}\\
&\qquad+\frac{1}{r^2}h_{\alpha\beta}g^{ij}g^{rs}\bar\nabla_{n+1}\mathcal{A}^\alpha_{ir}\bar\nabla_{n+1}\mathcal{A}^\alpha_{js} + \frac{2}{r^2}h_{\alpha\beta}g^{ij}g^{rs}\bar\nabla_r\mathcal{A}^\alpha_{in+1}\bar\nabla_s\mathcal{A}^\alpha_{jn+1}\\
&\qquad+\frac{4}{r^4}h_{\alpha\beta}g^{ij}\bar\nabla_{n+1}\mathcal{A}^\alpha_{in+1}\bar\nabla_{n+1}\mathcal{A}^\alpha_{jn+1} + \frac{2}{r^4}h_{\alpha\beta}g^{rs}\bar\nabla_r\mathcal{A}^\alpha_{n+1n+1}\bar\nabla_s\mathcal{A}^\alpha_{n+1n+1}\\
&\qquad+\frac{1}{r^6}h_{\alpha\beta}\bar\nabla_{n+1}\mathcal{A}^\alpha_{n+1n+1}\bar\nabla_{n+1}\mathcal{A}^\alpha_{n+1n+1}.
\end{aligned}$$

Note that in terms referring to the immersion F, latin indices are summed from 1 to $n+1$ whereas contractions due to the metric on c contain only summands from 1 to n:

$$\begin{aligned}
\bar g^{ij}\bar\nabla_k\mathcal{A}^\alpha_{ir}\bar\nabla_l\mathcal{A}^\alpha_{js} &= \sum_{1\leqslant i,j\leqslant n+1}\bar g^{ij}\bar\nabla_k\mathcal{A}^\alpha_{ir}\bar\nabla_l\mathcal{A}^\alpha_{js}\\
&= \sum_{1\leqslant i,j\leqslant n} g^{ij}\bar\nabla_k\mathcal{A}^\alpha_{ir}\bar\nabla_l\mathcal{A}^\alpha_{js} + \frac{1}{r^2}\bar\nabla_k\mathcal{A}^\alpha_{n+1r}\bar\nabla_l\mathcal{A}^\alpha_{n+1s}\\
&= g^{ij}\bar\nabla_k\mathcal{A}^\alpha_{ir}\bar\nabla_l\mathcal{A}^\alpha_{js} + \frac{1}{r^2}\bar\nabla_k\mathcal{A}^\alpha_{n+1r}\bar\nabla_l\mathcal{A}^\alpha_{n+1s}.
\end{aligned}$$

B.2. Evolution equations for manifolds under the perturbed flow

In the following we deduce some evolution equations for geometric quantities of the family of generating immersions $c_t(S^n) \subset \mathbb{R}^m_{>0}$ evolving under the perturbed flow:

$$\frac{\partial}{\partial t} c(x,t) = \mathcal{H} = H + \lambda \tag{B.5a}$$

$$c(x,0) = c_0(x), \tag{B.5b}$$

where $\lambda = -\frac{1}{r}\left(e_1^\perp\right)$, H denotes the mean curvature vector and A the second fundamental form of c. As above, ∇ refers to the induced connection on c. R is the Riemannian curvature tensor of c with local expression $R_{klij} = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right\rangle$.

Lemma B.1. *For a n-dimensional generating immersion c, we have*

$$|\nabla r|^2 = 1 - r^2|\lambda|^2, \tag{B.6a}$$

$$\Delta r = -rH^\alpha\lambda_\alpha, \tag{B.6b}$$

$$\nabla_l\lambda^\alpha = -\frac{1}{r}\frac{\partial r}{\partial x^l}\lambda^\alpha - A_l^{\beta m}\lambda_\beta c_m^\alpha + \frac{1}{r}\nabla^m r A_{lm}^\alpha. \tag{B.6c}$$

Proof. Since $\nabla_i r = \delta_{\alpha\beta}c_i^\alpha(e_1)^\beta$ and $e_1^\perp = e_1 - \left(\delta_{\gamma\beta}c_i^\gamma(e_1)^\beta g^{ij}c_j^\alpha\right)\frac{\partial}{\partial y^\alpha}$, the norm of $e_1^\perp$ is given by

$$r^2|\lambda|^2 = \left|e_1^\perp\right|^2 = \left|e_1 - \nabla_i r g^{ij}c_j^\alpha\frac{\partial}{\partial y^\alpha}\right|^2 = |e_1|^2 - |\nabla r|^2.$$

This is the first equation. The second equation follows easily since ∇ is a metric connection, c is immersed in euclidean space and by definition of λ:

$$\Delta r = \Delta(\langle c, e_1\rangle) = \langle\Delta c, e_1\rangle = \langle H, e_1\rangle = \left\langle H, e_1^\perp\right\rangle = -r\langle H,\lambda\rangle.$$

The first derivative of λ can be computed as follows

$$\begin{aligned}\nabla_l\lambda^\alpha &= -\frac{1}{r}\frac{\partial r}{\partial x^l}\lambda^\alpha - \frac{1}{r}\nabla_l e_1^\perp = -\frac{1}{r}\frac{\partial r}{\partial x^l}\lambda^\alpha + \frac{1}{r}\nabla_l\left(\nabla_m r g^{mn}c_n^\alpha\right)\\ &= -\frac{1}{r}\frac{\partial r}{\partial x^l}\lambda^\alpha + \frac{1}{r}\nabla_l\nabla^m r c_m^\alpha + \frac{1}{r}\nabla^m r A_{lm}^\alpha\\ &= -\frac{1}{r}\frac{\partial r}{\partial x^l}\lambda^\alpha - A_l^{\beta m}\lambda_\beta c_m^\alpha + \frac{1}{r}\nabla^m r A_{lm}^\alpha.\end{aligned} \tag{B.7}$$

The second covariant derivative of λ is given by

$$\begin{aligned}
\nabla_k\nabla_l\lambda^\alpha &= \left(\frac{1}{r^2}\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l} + A^\beta_{kl}\lambda_\beta\right)\lambda^\alpha - \frac{1}{r}\frac{\partial r}{\partial x^l}\nabla_k\lambda^\alpha - \left(\nabla_k A_l^{\beta m}\lambda_\beta + A_l^{\beta m}\nabla_k\lambda_\beta\right)c_m^\alpha \\
&\quad - A_l^{\beta m}\lambda_\beta A^\alpha_{km} - \left(\frac{1}{r^2}\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^m} + A^\beta_{km}\lambda_\beta\right)A_l^{\alpha m} + \frac{1}{r}\nabla^m r\nabla_k A^\alpha_{lm} \\
&= \left(\frac{2}{r^2}\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l} + A^\beta_{kl}\lambda_\beta\right)\lambda^\alpha \\
&\quad - \left(\nabla_k A_l^{\beta m}\lambda_\beta - \frac{1}{r}\frac{\partial r}{\partial x^l}A_k^{\beta m}\lambda_\beta - \frac{1}{r}\frac{\partial r}{\partial x^k}A_l^{\beta m}\lambda_\beta + \frac{1}{r}\frac{\partial r}{\partial x^n}A_l^{\beta m}A^n_{\beta k}\right)c_m^\alpha \\
&\quad - A_l^{\beta m}\lambda_\beta A^\alpha_{km} - \left(\frac{1}{r^2}\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^m} + A^\beta_{km}\lambda_\beta\right)A_l^{\alpha m} \\
&\quad - \frac{1}{r^2}\frac{\partial r}{\partial x^l}\frac{\partial r}{\partial x^m}A_k^{\alpha m} + \frac{1}{r}\nabla^m r\nabla_k A^\alpha_{lm}.
\end{aligned} \tag{B.8}$$

Taking the trace gives

$$\begin{aligned}
\Delta\lambda^\alpha &= \left(\frac{2}{r^2}\left|\nabla r\right|^2 + H^\beta\lambda_\beta\right)\lambda^\alpha \\
&\quad - \left(\nabla^m H^\beta\lambda_\beta - g^{kl}\frac{2}{r}\frac{\partial r}{\partial x^l}A_k^{\beta m}\lambda_\beta + g^{kl}\frac{1}{r}\frac{\partial r}{\partial x^n}A_l^{\beta m}A^n_{\beta k}\right)c_m^\alpha \\
&\quad - 2g^{kl}A_l^{\beta m}\lambda_\beta A^\alpha_{km} - 2\frac{1}{r^2}\frac{\partial r}{\partial x^n}\frac{\partial r}{\partial x^m}A^{\alpha mn} + \frac{1}{r}\nabla^m r\nabla_m H^\alpha + \frac{1}{r}\nabla^m r c_n^\alpha R_m^n.
\end{aligned} \tag{B.9}$$

For space curves this equations simplify to the stated equations in Lemma 3.1. □

One easily gets

Lemma B.2. *For a family of n dimensional manifolds C_t evolving under* (B.5) *we have*

$$\begin{aligned}
\frac{\partial}{\partial t}r &= \langle H + \lambda, e_1\rangle = \Delta r - r\left|\lambda\right|^2 \\
\frac{\partial}{\partial t}c_i^\alpha &= \nabla_i\left(H^\alpha + \lambda^\alpha\right) = \Delta c_i^\alpha - R_i^m c_m^\alpha + \nabla_i\lambda^\alpha, \\
\frac{\partial}{\partial t}g_{ij} &= -2A^\alpha_{ij}\left(H_\alpha + \lambda_\alpha\right) =: -2\left(a_{ij} + l_{ij}\right)
\end{aligned}$$

with $a_{ij} := H_\alpha A^\alpha_{ij}$ and $l_{ij} := A^\alpha_{ij}\lambda_\alpha$.

By standard computations we derive the evolution equation for the second fundamental form and mean curvature vector. First we need

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^k_{ij} &= \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial t}g_{il,j} + \frac{\partial}{\partial t}g_{jl,i} - g_{ij,l}\right) = \frac{1}{2}g^{kl}\left(\nabla_j\frac{\partial}{\partial t}g_{il} + \nabla_i\frac{\partial}{\partial t}g_{jl} - \nabla_l\frac{\partial}{\partial t}g_{ij}\right) \\
&= -\left(\nabla_i a_j^k + \nabla_j a_i^k - \nabla^k a_{ij}\right) - \left(\nabla_i l_j^k + \nabla_j l_i^k - \nabla^k l_{ij}\right) \\
&= -\left(\nabla_i a_j^k + \nabla_j a_i^k - \nabla^k a_{ij}\right) - \left(A_j^{\alpha k}\nabla_i\lambda_\alpha + A_i^{\alpha k}\nabla_j\lambda_\alpha - A^\alpha_{ij}\nabla^k\lambda_\alpha\right) - \nabla^k A^\alpha_{ij}\lambda_\alpha.
\end{aligned}$$

Let $b_{ij} := A_i^{\alpha m} A_{\alpha m j}$, then

$$\begin{aligned}
\frac{\partial}{\partial t} A_{ij}^{\alpha} =& \nabla_i \frac{\partial}{\partial t} c_j^{\alpha} - c_k^{\alpha} \frac{\partial}{\partial t} \Gamma_{ij}^k = \nabla_i \nabla_j H^{\alpha} + \nabla_i \nabla_j \lambda^{\alpha} - c_k^{\alpha} \frac{\partial}{\partial t} \Gamma_{ij}^k \\
=& \Delta A_{ij}^{\alpha} + 2A^{\alpha mn} R_{minj} - A_{mi}^{\alpha} R_j^m - A_{mj}^{\alpha} R_i^m + \nabla_i \nabla_j \lambda^{\alpha} \\
&+ c_k^{\alpha} \left(\nabla_i b_j^k + \nabla_j b_i^k - \nabla^k b_{ij} + \nabla_i l_j^k + \nabla_j l_i^k - \nabla^k l_{ij} \right) \\
=& \Delta A_{ij}^{\alpha} + 2A^{\alpha mn} R_{minj} - A_{mi}^{\alpha} R_j^m - A_{mj}^{\alpha} R_i^m + \left(\frac{2}{r^2} \frac{\partial r}{\partial x^k} \frac{\partial r}{\partial x^l} + l_{kl} \right) \lambda^{\alpha} \\
&- l_l^m A_{km}^{\alpha} - \left(\frac{1}{r^2} \frac{\partial r}{\partial x^k} \frac{\partial r}{\partial x^m} + l_{km} \right) A_l^{\alpha m} + \frac{1}{r} \nabla^m r \nabla_k A_{lm}^{\alpha} + \text{ tangential terms}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t} |A|^2 =& 2A_{\alpha ij} \frac{\partial}{\partial t} A^{\alpha ij} + 2 \frac{\partial}{\partial t} g^{ij} A_i^{\alpha m} A_{\alpha im} \\
=& \Delta |A|^2 - 2 |\nabla_i A_{kl}^{\alpha}|^2 + 4A_{\alpha}^{ij} A^{\alpha mn} R_{minj} - 4b_{ij} R^{ij} + 2A_{\alpha}^{ij} \nabla_i \nabla_j \lambda^{\alpha} \\
&+ 2 \frac{\partial}{\partial t} g^{ij} A_i^{\alpha m} A_{\alpha jm} \\
=& \Delta |A|^2 - 2 |\nabla_i A_{kl}^{\alpha}|^2 + 4A_{\alpha}^{ij} A^{\alpha mn} R_{minj} - 4b_{ij} R^{ij} + 2A_{\alpha}^{ij} \nabla_i \nabla_j \lambda^{\alpha} \\
&+ 4a^{ij} A_i^{\alpha m} A_{\alpha jm} + 4A^{\beta ij} \lambda_{\beta} A_i^{\alpha m} A_{\alpha jm} \\
=& \Delta |A|^2 - 2 |\nabla_i A_{kl}^{\alpha}|^2 + 4A_{\alpha}^{ij} A^{\alpha mn} R_{minj} - 4b_{ij} R^{ij} + 2A_{\alpha}^{ij} \nabla_i \nabla_j \lambda^{\alpha} \\
&+ 4a^{ij} b_{ij} + 4A^{\beta ij} \lambda_{\beta} b_{ij}.
\end{aligned} \tag{B.10}$$

From the evolution equation for the second fundamental form, we get for a n-dimensional generating manifold

$$\begin{aligned}
\frac{\partial}{\partial t} H^{\alpha} =& g^{kl} \frac{\partial}{\partial t} A_{kl}^{\alpha} + \frac{\partial}{\partial t} g^{kl} A_{kl}^{\alpha} \\
=& \Delta H^{\alpha} + \Delta \lambda^{\alpha} + 2 \left(a^{kl} + l^{kl} \right) A_{kl}^{\alpha} + c_n^{\alpha} \left(2\nabla^m b_m^n - \nabla^n |A|^2 + 2\nabla^m l_m^n - \nabla^n \left(H^{\beta} \lambda_{\beta} \right) \right)
\end{aligned} \tag{B.11}$$

and so

$$\frac{\partial}{\partial t} |H|^2 = \Delta |H|^2 - 2 |\nabla H|^2 + 2\Delta \lambda^{\alpha} H_{\alpha} + 4a^{kl} a_{kl} + 4l^{kl} a_{kl}. \tag{B.12}$$

After computing the flow equation for $V := e_1^{\perp}$

$$\begin{aligned}
\frac{\partial}{\partial t} V^{\alpha} = \Delta V^{\alpha} &+ 2\nabla^n \nabla^m r A_{mn}^{\alpha} - \frac{1}{r} \nabla^n r \nabla^m r A_{mn}^{\alpha} + \frac{1}{r} |\nabla r|^2 \lambda^{\alpha} \\
&+ c_m^{\alpha} \left(2R^{mn} \nabla_n r - \nabla^m r |\lambda|^2 + \nabla_n r l^{nm} - 2a^{mn} \nabla_n r \right),
\end{aligned}$$

one deduces

$$\frac{\partial}{\partial t}\lambda^\alpha = \Delta\lambda^\alpha + \lambda^\alpha\left(|\lambda|^2 - \frac{3}{r^2}|\nabla r|^2\right) - \frac{2}{r}\nabla^n\nabla^m r A^\alpha_{mn} \\ + \frac{3}{r^2}\nabla^n r\nabla^m r A^\alpha_{mn} - \frac{1}{r}c^\alpha_m\left(2R^{mn}\nabla_n r - \nabla^m r|\lambda|^2 + 3\nabla_n r l^{nm} - 2a^{mn}\nabla_n r\right).$$

In the special case that $n = 1$, i. e. c is a curve, these two equations simplify to

$$\frac{\partial}{\partial t}V^\alpha = \Delta V^\alpha + 2\Delta r H^\alpha - \frac{1}{r}|\nabla r|^2 H^\alpha + \frac{1}{r}|\nabla r|^2\lambda^\alpha \\ + c^\alpha_m\left(-\nabla^m r|\lambda|^2 + \nabla^m r H^\alpha\lambda_\alpha - 2|H|^2\nabla^m r\right)$$

and

$$\frac{\partial}{\partial t}\lambda^\alpha = \Delta\lambda^\alpha + \lambda^\alpha\left(|\lambda|^2 - \frac{3}{r^2}|\nabla r|^2\right) - \frac{2}{r}\Delta r H^\alpha \\ + \frac{3}{r^2}|\nabla r|^2 H^\alpha - \frac{1}{r}c^\alpha_m\left(-\nabla^m r|\lambda|^2 + 3\nabla^m r H^\alpha\lambda_\alpha - 2|H|^2\nabla^m r\right). \tag{B.13}$$

B.3. Further evolutions equations for a space curve under the perturbed flow

In Lemma 4.15, the maximal total loss of the torsion is bounded on some small interval of time and space. Therefore, it is necessary to compute the evolution equation of $\tau_i = \nabla_i\nu^\alpha B_\alpha$ away from where the curvature $|H|$ vanishes.

Lemma B.3. *The evolution equation for the unit normal and binormal vector field of space curves evolving under the perturbed flow* (3.5)

$$\frac{\partial}{\partial t}\nu^\alpha = \Delta\nu^\alpha + B^\alpha\tau_m\left(\frac{2}{|H|}\nabla^m|H| + \frac{1}{r}\nabla^m r\right) + \lambda^\alpha\left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle\nu,\lambda\rangle\right) \\ + \nu^\alpha\left(|H|^2 + |\tau|^2 - \langle\nu,\lambda\rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2\langle\nu,\lambda\rangle\right) \\ + c^\alpha_m\left(-|H|\frac{1}{r}\nabla^m r - \frac{1}{r}\nabla^m r\langle\nu,\lambda\rangle + \frac{2}{|H|}\frac{1}{r}\nabla^m r|\lambda|^2\right), \tag{B.14}$$

$$\frac{\partial}{\partial t}B^\alpha = \Delta B^\alpha + |\tau|^2 B^\alpha - 2c^\alpha_m\tau^m|H| + c^\alpha_m g^{mn}\frac{1}{r}\nabla_n r\langle B,\lambda\rangle \\ - \nu^\alpha\langle B,\lambda\rangle\left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle\nu,\lambda\rangle\right) - \nu^\alpha\tau_m\left(\frac{2}{|H|}\nabla^m|H| + \frac{1}{r}\nabla^m r\right). \tag{B.15}$$

Proof. From (3.12) we get

$$
\begin{aligned}
2|H|\frac{\partial}{\partial t}|H| = \frac{\partial}{\partial t}|H|^2 =& 2|H|\Delta|H| + 2|\nabla|H||^2 - 2|\nabla H|^2 + \frac{1}{r}\nabla^m r \nabla_m |H|^2 \\
&+ 2\langle H,\lambda\rangle\left(\frac{2}{r^2}|\nabla r|^2 + \langle H,\lambda\rangle\right) + 4|H|^2\left(|H|^2 - \frac{1}{r}|\nabla r|^2\right) \\
=& 2|H|\Delta|H| - 2|H|^2\left(|H|^2 + |\tau|^2\right) + \frac{2}{r}|H|\nabla^m r\nabla_m|H| \\
&+ 2\langle H,\lambda\rangle\left(\frac{2}{r^2}|\nabla r|^2 + \langle H,\lambda\rangle\right) + 4|H|^2\left(|H|^2 - \frac{1}{r}|\nabla r|^2\right)
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{\partial}{\partial t}|H| =& \Delta|H| - |H|\left(|H|^2 + |\tau|^2\right) + \frac{1}{r}\nabla^m r\nabla^m|H| \\
&+ \langle \nu,\lambda\rangle\left(\frac{2}{r^2}|\nabla r|^2 + \langle H,\lambda\rangle\right) + 2|H|\left(|H|^2 - \frac{1}{r^2}|\nabla r|^2\right) \\
=& \Delta|H| + \frac{1}{r}\nabla^m r\nabla^m|H| + |H|\left(|H|^2 - |\tau|^2 + \langle\nu,\lambda\rangle^2 - \frac{2}{r}|\nabla r|^2\right) + \frac{2}{r^2}|\nabla r|^2\langle\nu,\lambda\rangle
\end{aligned}
$$

So with (3.11) we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\nu^\alpha =& \frac{1}{|H|}\frac{\partial}{\partial t}H^\alpha - \frac{1}{|H|}\frac{\partial}{\partial t}(|H|)\,\nu^\alpha \\
=& \frac{1}{|H|}\Bigg(|H|\Delta\nu^\alpha + 2\nabla^m|H|\nabla_m\nu^\alpha + \nu^\alpha\Delta|H| \\
&+\nu^\alpha\left(2|H||H|^2 - |H|\frac{2}{r^2}|\nabla r|^2 + \frac{1}{r}\nabla^m r\nabla_m|H|\right) \\
&+\lambda^\alpha\left(\frac{2}{r^2}|\nabla r|^2 + |H|\langle\nu,\lambda\rangle\right) + \frac{1}{r}\nabla^m r|H|\tau_m B^\alpha \\
&+c_m^\alpha\left(-|H|^2\frac{1}{r}\nabla^m r + \nabla^m|H|^2 - \nabla^m|\lambda|^2 + \frac{1}{r}\nabla^m r\langle H,\lambda\rangle\right)\Bigg) \\
&-\frac{1}{|H|}\Bigg(\Delta|H| + \frac{1}{r}\nabla^m r\nabla^m|H| + \frac{2}{r^2}|\nabla r|^2\langle\nu,\lambda\rangle \\
&+|H|\left(|H|^2 - |\tau|^2 + \langle\nu,\lambda\rangle^2 - \frac{2}{r}|\nabla r|^2\right)\Bigg)\nu^\alpha.
\end{aligned}
$$

The Frenet equations (3.13) and $\nabla_m|\lambda|^2 = \frac{2}{r}\nabla_m r\left(\langle H,\lambda\rangle - |\lambda|^2\right)$ imply

$$
\begin{aligned}
\frac{\partial}{\partial t}\nu^\alpha =& \Delta\nu^\alpha + B^\alpha\tau_m\left(\frac{2}{|H|}\nabla^m|H| + \frac{1}{r}\nabla^m r\right) + \lambda^\alpha\left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle\nu,\lambda\rangle\right) \\
&+\nu^\alpha\left(|H|^2 + |\tau|^2 - \langle\nu,\lambda\rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2\langle\nu,\lambda\rangle\right) \\
&+c_m^\alpha\left(-|H|\frac{1}{r}\nabla^m r - \frac{1}{r}\nabla^m r\langle\nu,\lambda\rangle + \frac{2}{|H|}\frac{1}{r}\nabla^m r|\lambda|^2\right).
\end{aligned}
$$

Furthermore, with

$$\frac{\partial}{\partial t}c_i^\alpha = \nabla_i H^\alpha + \nabla_i \lambda^\alpha = \Delta c_i^\alpha - \frac{1}{r}\nabla_i r \lambda^\alpha - H^\beta \lambda_\beta c_i^\alpha + \frac{1}{r}\nabla_i r H^\alpha$$

and

$$\frac{\partial}{\partial t}\nu^\alpha \nu_\alpha = 0, \quad \nabla_i \nu^\alpha \nu_\alpha = 0, \quad \frac{\partial}{\partial t}B^\alpha B_\alpha = 0, \quad \nabla_i B^\alpha B_\alpha = 0,$$

one computes

$$\begin{aligned}\frac{\partial}{\partial t}B^\alpha =& - B_\beta \frac{\partial}{\partial t}c_m^\beta g^{mn} c_n^\alpha - B_\beta \frac{\partial}{\partial t}\nu^\beta \nu^\alpha \\ =& - c_m^\alpha g^{mn} B_\beta \left(\Delta c_n^\beta - \frac{1}{r}\nabla_n r \lambda^\beta\right) - \nu^\alpha B_\beta \left(\Delta \nu^\beta + \lambda^\beta \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle \nu, \lambda\rangle\right)\right) \\ & - \nu^\alpha \tau_m \left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right).\end{aligned}$$

Reordering and adding appropriate terms give

$$\begin{aligned}\frac{\partial}{\partial t}B^\alpha =& \underbrace{\Delta B^\beta B_\beta B^\alpha + |\nabla B|^2 B^\alpha}_{=0} - c_m^\alpha g^{mn} B_\beta \Delta c_n^\beta \underbrace{-2 c_m^\alpha g^{mn} \nabla^k B_\beta \nabla_k c_n^\beta - 2 c_m^\alpha \tau^m |H|}_{=0} \\ & - \nu^\alpha \Delta \nu^\beta B_\beta - \underbrace{2\nu^\alpha \nabla^m \nu^\beta \nabla_m B_\beta}_{=0} + c_m^\alpha g^{mn} \frac{1}{r}\nabla_n r \langle B, \lambda\rangle \\ & - \nu^\alpha \langle B, \lambda\rangle \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle \nu, \lambda\rangle\right) - \nu^\alpha \tau_m \left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right) \\ =& \Delta B^\alpha + |\tau|^2 B^\alpha - 2 c_m^\alpha \tau^m |H| + c_m^\alpha g^{mn}\frac{1}{r}\nabla_n r \langle B, \lambda\rangle \\ & - \nu^\alpha \langle B, \lambda\rangle \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle \nu, \lambda\rangle\right) - \nu^\alpha \tau_m \left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right).\end{aligned}$$

□

Lemma B.4. *The evolution equation of the torsion is*

$$\begin{aligned}\frac{\partial}{\partial t}\tau_i =& \Delta \tau_i + \nabla_i \tau_m \left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right) + \tau_m \nabla_i \left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right) \\ & + \tau_i \left(|H|^2 - \langle \nu, \lambda\rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 \langle \nu, \lambda\rangle + \langle B, \lambda\rangle^2\right) \\ & - \frac{1}{r}\nabla_i r \langle B, \lambda\rangle \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + 2\langle \nu, \lambda\rangle\right) + \lambda^\alpha B_\alpha \nabla_i \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2\right),\end{aligned} \tag{B.16}$$

Proof. Since

$$
\begin{aligned}
\frac{\partial}{\partial t}\tau_i =& \frac{\partial}{\partial t}\left(\nabla_i \nu^\alpha\right) B_\alpha + \nabla_i \nu^\alpha \frac{\partial}{\partial t} B_\alpha = \nabla_i \left(\frac{\partial}{\partial t}\nu^\alpha\right) B_\alpha + \nabla_i \nu^\alpha \frac{\partial}{\partial t} B_\alpha \\
=& \nabla_i \left(\frac{\partial}{\partial t}\nu^\alpha B_\alpha\right) - \frac{\partial}{\partial t}\nu^\alpha \nabla_i B_\alpha + \nabla_i \nu^\alpha \frac{\partial}{\partial t} B_\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta \nu^\alpha = \nabla^m \left(- |H| c_m^\alpha + \tau_m B^\alpha\right), \quad \nu_\alpha \Delta \nu^\alpha = - |H|^2 - |\tau|^2, \\
&\Delta B^\alpha = \nabla^m \left(-\tau_m \nu^\alpha\right), \quad \nabla_i \nu^\alpha \Delta B_\alpha = -\tau_i \left(|\tau|^2 + |H|^2\right) = -\Delta \nu_\alpha \nabla_i B^\alpha,
\end{aligned}
$$

the evolution equation of the torsion form is given by

$$
\begin{aligned}
\frac{\partial}{\partial t}\tau_i =& \nabla_i \left(\Delta \nu^\alpha B_\alpha + \tau_m \left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right) + \lambda^\alpha B_\alpha \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle \nu, \lambda\rangle\right)\right) \\
&+ \tau_i \left(\nu_\alpha \Delta \nu^\alpha + |H|^2 + |\tau|^2 - \langle \nu, \lambda\rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 \langle \nu, \lambda\rangle\right) \\
&+ \tau_i \nu_\alpha \lambda^\alpha \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle \nu, \lambda\rangle\right) + \nabla_i \nu^\alpha \Delta B_\alpha \\
&+ \tau_i |\tau|^2 + 2\tau_i |H|^2 - |H| \frac{1}{r}\nabla_i r \langle B, \lambda\rangle .
\end{aligned}
$$

Adding some appropriate zero terms leads to

$$
\begin{aligned}
\frac{\partial}{\partial t}\tau_i =& \Delta \tau_i - 2\nabla^m \nabla_i \nu^\alpha \nabla_m B_\alpha \underbrace{+\Delta \nu^\alpha \nabla_i B_\alpha + \tau_i \nu^\alpha \Delta \nu_\alpha}_{=0} + \nabla_i \tau^m \left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right) \\
&+ \tau^m \nabla_i \left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right) - \tau_i \nu_\alpha \lambda^\alpha \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle \nu, \lambda\rangle\right) \\
&- \frac{1}{r}\nabla_i r \lambda^\alpha B_\alpha \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle \nu, \lambda\rangle + |H|\right) + \lambda^\alpha B_\alpha \nabla_i \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle \nu, \lambda\rangle\right) \\
&+ \tau_i \left(3|H|^2 + 2|\tau|^2 - \langle \nu, \lambda\rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 \langle \nu, \lambda\rangle\right) \\
&+ \tau_i \nu_\alpha \lambda^\alpha \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle \nu, \lambda\rangle\right),
\end{aligned}
$$

which simplifies to

$$\begin{aligned}
\frac{\partial}{\partial t}\tau_i =& \Delta\tau_i - 2\tau_i\left(|H|^2 + |\tau|^2\right) + \nabla_i\tau_m\left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right)\\
&+ \tau_m\nabla_i\left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right)\\
&+ \tau_i\left(3|H|^2 + 2|\tau|^2 - \langle\nu,\lambda\rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2\langle\nu,\lambda\rangle\right)\\
&- \frac{1}{r}\nabla_i r\lambda^\alpha B_\alpha\left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle\nu,\lambda\rangle + |H|\right) + \lambda^\alpha B_\alpha\nabla_i\left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \langle\nu,\lambda\rangle\right)\\
=& \Delta\tau_i + \nabla_i\tau_m\left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right) + \tau_m\nabla_i\left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right)\\
&+ \tau_i\left(|H|^2 - \langle\nu,\lambda\rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2\langle\nu,\lambda\rangle + \langle\lambda,B\rangle^2\right)\\
&- \frac{1}{r}\nabla_i r\lambda^\alpha B_\alpha\left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + 2\langle\nu,\lambda\rangle\right) + \lambda^\alpha B_\alpha\nabla_i\left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2\right).
\end{aligned}$$

□

Hence, the evolution equation of the norm computes to

$$\begin{aligned}
\frac{\partial}{\partial t}|\tau|^2 =& 2\tau^i\Delta\tau_i + 2\tau^i\nabla_i\tau_m\left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right) + 2\tau^i\tau_m\nabla_i\left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right)\\
&- 2\tau^i\frac{1}{r}\nabla_i r\lambda^\alpha B_\alpha\left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + 2\langle\nu,\lambda\rangle\right) + 2\tau^i\lambda^\alpha B_\alpha\nabla_i\left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2\right)\\
&+ 2|\tau|^2\left(2|H|^2 - \langle\nu,\lambda\rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2\langle\nu,\lambda\rangle + \langle\lambda,B\rangle^2 + H^\alpha\lambda_\alpha\right),
\end{aligned} \tag{B.17}$$

where the term $2|\tau|^2\left(|H|^2 + \langle H,\lambda\rangle\right)$ comes from the derivation of the metric.

C. Rotationally symmetric surfaces in arbitrary dimensional euclidean space

It is a straightforward calculation to generalize the results from Chapter 4 to surfaces in arbitrary dimensional euclidean space $\mathbb{R}^{m+1}$, $m \geqslant 3$. As in [44], the only difference is, that there is not only one torsion form but $(m-2)$ many: $\tau_1, \dots, \tau_{m-2}$.

Let $c_t : S^1 \to \mathbb{R}^m_{>0}$, $m \geqslant 3$, be any family of immersed curves evolving under the perturbed mean curvature flow (3.5):

$$\frac{\partial}{\partial t} c = H + \lambda,$$
$$c(\cdot, 0) = c_0,$$

where H is the mean curvature vector of the curve c_t and $\lambda = -\frac{1}{r} e_1^\perp$. Then each c_t generates an immersed surface

$$F_t\left(S^1 \times S^1\right) := \left\{\left(c^1\left(x^1\right)\cos x^2, c^2\left(x^1\right), \dots, c^m\left(x^1\right), c^1\left(x^1\right)\sin x^2\right) \in \mathbb{R}^{m+1}; \; x^1, x^2 \in S^1\right\}$$

such that the family (F_t) evolves under the mean curvature flow.

In this section, it is explained that the property of becoming planar in the singularity does not depend on the dimension of the surrounding space:

Lemma C.1. *Let c_t be a solution to the perturbed flow which develops a singularity away from the plane of rotation, i. e. $r \geqslant r_0$ on the maximal time interval $[0, T)$. Then the rescaled solutions converge along a subsequence of any essential blow-up sequence (p_n, t_n) to a family of convex, planar curves c_∞ moving by mean curvature flow.*

The proof is more or less the same as for curves in $\mathbb{R}^3$, and the differences are outlined in the following.

Since the dilation-invariant estimates (cf. Proposition 4.5, Corollary 4.6) are also valid in arbitrary dimension, these estimates imply the long-time existence of the solution as long as the curvature and the perturbation term λ remain bounded (cf. Proposition 4.7).

The first step is to bound the total loss of the curvature and the first torsion form. Therefore, assume that c_t is a Frenet curve, i. e. all higher derivatives up to order m $\frac{\partial c^\alpha}{\partial x^1}\frac{\partial}{\partial y^\alpha}, \dots, \frac{\partial^m c^\alpha}{\partial x^{1m}}\frac{\partial}{\partial y^\alpha}$ are linear independent. Then choose the unique Frenet frame $T, B_0, B_1, \dots, B_{m-2}$ such that T is a unit tangent vector, $T, B_0, B_1, \dots, B_{m-2}$ is a positive oriented orthonormal Basis of $\mathbb{R}^m$, and the orientation of $T, B_0, \dots, B_k$ is the same as the one of $\frac{\partial c^\alpha}{\partial x^1}\frac{\partial}{\partial y^\alpha}, \dots, \frac{\partial^k c^\alpha}{\partial (x^1)^k}\frac{\partial}{\partial y^\alpha}$ for all $0 \leqslant k \leqslant m-2$. Let

$$\tau_{I+1\,i} := \delta_{\alpha\beta}\nabla_i B_I^\alpha B_{I+1}^\beta \text{ for } i = 1,\ I = 0, \dots, m-3 \tag{C.1}$$

be the Ith torsion 1-form. From now on, we will write ν instead of B_0. For more details and a proof of the following Lemma see for example [61].

Lemma C.2. *For $I = 0, \dots, m-2$ the Frenet equations are:*

$$\nabla_i c_j^\alpha = |H|\, g_{ij}\nu^\alpha, \quad \nabla_i \nu^\alpha = -|H|\, c_i^\alpha + \tau_{1\,i} B_1^\alpha, \quad \nabla_i B_I^\alpha = -\tau_{I-1\,i} B_{I-1}^\alpha + \tau_{I+1\,i} B_{I+1}^\alpha, \tag{C.2}$$

where $\tau_{m-1} = 0$.

Lemma 3.6 implies that it suffices to control the loss of the curvature and the first torsion form and to show that the first torsion form vanishes in the singularity. Whenever c is a Frenet curve, the first covariant derivatives of the tangent vector are given in terms of the curvature and the torsion forms:

$$\left|\nabla_i c_j^\alpha\right| = |H|^2, \tag{C.3a}$$

$$|\nabla_k \nabla_l c_i^\alpha|^2 = |\nabla |H||^2 + |H|^4 + |H|^2 |\tau_1|^2 \tag{C.3b}$$

$$\begin{aligned} |\nabla_j \nabla_i \nabla_k c_l^\alpha|^2 &= \left|\nabla_j \nabla_i |H| - |H|^3 - |H|\, \tau_{1\,i}\tau_{1\,j}\right|^2 + \left|3\,|H|\, \nabla_j |H|\right|^2 \\ &\quad + \left|2\nabla_j |H|\, \tau_{1\,i} + |H|\, \nabla_j \tau_{1\,i}\right|^2 + |H|^2 |\tau_1|^2 |\tau_2|^2, \end{aligned} \tag{C.3c}$$

$$\begin{aligned} &|\nabla_m \nabla_j \nabla_i \nabla_k c_l^\alpha|^2 \\ &\quad = \Big|3\nabla_m \nabla_j |H|\, \tau_{1\,i} + 3\nabla_j |H|\, \nabla_m \tau_{1\,i} + |H|\, \nabla_j \nabla_m \tau_{1\,i} - |H|^3 g_{ij}\tau_{1\,m} \\ &\qquad - |H|\, \tau_{1\,m}\tau_{1\,i}\tau_{1\,j} - |H|\, \tau_{1\,m}\tau_{2\,i}\tau_{2\,j}\Big|^2 \\ &\qquad + \big(\text{terms involving } \tau_1, \tau_2, \tau_3, |H|, \nabla |H|, \nabla\tau_1, \nabla\tau_2, \nabla^2 |H|, \nabla^3 |H|\big)^2. \end{aligned} \tag{C.3d}$$

As in Chapter 4, the summands can be bounded on small time intervals by using the dilation-invariant estimates (cp. Corollary 4.8):

Corollary C.3. *Let $t_n := \tilde{t_n} + \frac{1}{4\sigma\mathcal{M}_{\tilde{t_n}}}$ for some $\tilde{t_n} \in [0, t_n)$. Assume that $\rho\mathcal{M}_t \leqslant \mathcal{M}_{t_n}$ for all $t \leqslant t_n$. Then there exist constants $\tilde{c}_1, \dots, \tilde{c}_4 < \infty$, depending only on ρ, such that for $t \in \left[t_n, t_n + \frac{\rho}{4\sigma\mathcal{M}_{t_n}}\right]$ the following bounds are valid*

$$\begin{aligned} &|H|^2 \leqslant \tilde{c}_1\mathcal{M}_{t_n},\ |\nabla |H||^2 \leqslant \tilde{c}_2{\mathcal{M}_{t_n}}^2,\ |H|^2 |\tau_1|^2 \leqslant \tilde{c}_2{\mathcal{M}_{t_n}}^2,\ |H|^2 |\tau_1|^2 |\tau_2|^2 \leqslant \tilde{c}_3{\mathcal{M}_{t_n}}^3, \\ &\left|\nabla_j \nabla_i |H| - |H|^3 g_{ij} - |H|\, \tau_{1\,i}\tau_{1\,j}\right|^2 \leqslant \tilde{c}_3{\mathcal{M}_{t_n}}^3, \quad |2\nabla_j |H|\, \tau_{1\,i} + |H|\, \nabla_j \tau_{1\,i}|^2 \leqslant \tilde{c}_3{\mathcal{M}_{t_n}}^3, \\ &\Big|3\nabla_m \nabla_j |H|\, \tau_{1\,i} + 3\nabla_j |H|\, \nabla_m \tau_{1\,i} + |H|\, \nabla_j \nabla_m \tau_{1\,i} - |H|^3 g_{ij}\tau_{1\,m} \\ &\qquad\qquad - |H|\, \tau_{1\,m}\tau_{1\,i}\tau_{1\,j} - |H|\, \tau_{1\,m}\tau_{2\,i}\tau_{2\,j}\Big|^2 \leqslant \tilde{c}_4{\mathcal{M}_{t_n}}^4. \end{aligned}$$

Note, that we get bounds in terms of $\mathcal{N}_{t_n}$, if $\mathcal{N}_{t_n} \geqslant \frac{1}{r_0^2}$.

These estimates give control on the dissipation of the curvature $|H|$ such that Lemma 4.12 is converted at its face value to curves in higher dimensional euclidean space. Furthermore, the evolution equation of the first torsion form given below implies that its total loss is not too big, too, and a Lemma 4.15 is valid in this setting for τ_1.

The evolution equation of $\nabla_i \nu^\alpha$ is (cf. (B.14))

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla_i \nu^\alpha =& \nabla_i \frac{\partial}{\partial t}\nu^\alpha \\
=& \Delta \nabla_i \nu^\alpha + \lambda^\alpha \left(-\frac{1}{r}\nabla_i r + \nabla_i \left(\frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 + \nu^\beta \lambda_\beta\right)\right) - c_i^\alpha \left(|H|^3 + |H| |\tau_1|^2\right) \\
&+ c_m^\alpha \nabla_i \left(-|H| \frac{1}{r}\nabla^m r - \frac{1}{r}\nabla^m r \langle \nu, \lambda \rangle + \frac{2}{|H|}\frac{1}{r}\nabla^m r |\lambda|^2\right) \\
&+ B_1^\alpha \tau_{1\,i} \left(|H|^2 + |\tau_1|^2 - \langle \nu, \lambda \rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 \nu^\beta \lambda_\beta\right) \\
&+ B_1^\alpha \nabla_i \left(\tau_{1\,m}\left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right)\right) + \text{ terms in } \nu \text{ and } B_2.
\end{aligned}$$

Differentiating $|\tau_1|^2 + |H|^2 = g^{ij}\nabla_i \nu^\alpha \nabla_j \nu_\alpha$ give

$$\begin{aligned}
\frac{\partial}{\partial t}|\tau_1|^2 =& 2\frac{\partial}{\partial t}\left(\nabla^i \nu_\alpha\right)\left(-|H| c_i^\alpha + \tau_{1\,i} B_1^\alpha\right) + 2\left(|H|^2 + \langle H, \lambda \rangle\right)\left(|\tau_1|^2 + |H|^2\right) - \frac{\partial}{\partial t}|H|^2 \\
=& \Delta |\tau_1|^2 + \Delta |H|^2 - 2\nabla_m \nabla_i \nu^\alpha \nabla^m \nabla^i \nu_\alpha + 2|H|^4 + 2|H|^2 |\tau_1|^2 \\
&- 2|H| \nabla_i \left(-|H| \frac{1}{r}\nabla^i r - \frac{1}{r}\nabla^i r \langle \nu, \lambda \rangle + \frac{2}{|H|}\frac{1}{r}\nabla^i r |\lambda|^2\right) \\
&+ 2|\tau_1|^2 \left(|H|^2 + |\tau_1|^2 - \langle \nu, \lambda \rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 \nu^\beta \lambda_\beta\right) \\
&+ 2\tau_{1\,i}\nabla^i \left(\tau_{1\,m}\left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right)\right) \\
&+ 2\left(|H|^2 + \langle H, \lambda \rangle\right)\left(|\tau_1|^2 + |H|^2\right) - \frac{\partial}{\partial t}|H|^2 .
\end{aligned}$$

Using the equality $|\nabla\nabla\nu|^2 = |\nabla |H||^2 + |\nabla \tau_1|^2 + |H|^4 + |\tau_1|^4 + 2|H|^2 |\tau_1|^2 + |\tau_1|^2 |\tau_2|^2$, we get

$$\begin{aligned}
\frac{\partial}{\partial t}|\tau_1|^2 =& \Delta |\tau_1|^2 + 2|\tau_1|^2 \left(2|H|^2 - |\tau_2|^2\right) + 2\tau_{1\,i}\nabla^i \left(\tau_{1\,m}\left(\frac{2}{|H|}\nabla^m |H| + \frac{1}{r}\nabla^m r\right)\right) \\
&- 2|\nabla \tau_1|^2 + 2|\tau_1|^2 \left(\langle H, \lambda \rangle - \langle \nu, \lambda \rangle^2 - \frac{1}{|H|}\frac{2}{r^2}|\nabla r|^2 \nu^\beta \lambda_\beta\right) \\
&- \frac{\partial}{\partial t}|H|^2 + \Delta |H|^2 - 2|\nabla |H||^2 + 2|H|^2 \left(|H|^2 - |\tau_1|^2\right) + \frac{1}{r}\nabla^i r \nabla_i |H|^2 \\
&- 2|H|^2 \frac{1}{r^2}|\nabla r|^2 - 2|H| \nabla_i \left(-\frac{1}{r}\nabla^i r \langle \nu, \lambda \rangle + \frac{2}{|H|}\frac{1}{r}\nabla^i r |\lambda|^2\right) + 2|H|^2 \langle H, \lambda \rangle .
\end{aligned}$$

The evolution equation of $|H|^2$ (cf. (3.12)) gives

$$\begin{aligned}\frac{\partial}{\partial t}\left|\tau_1\right|^2 =&\Delta\left|\tau_1\right|^2+2\left|\tau_1\right|^2\left(2\left|H\right|^2-\left|\tau_2\right|^2\right)+2\tau_{1\,i}\nabla^i\left(\tau_{1\,m}\left(\frac{2}{|H|}\nabla^m\left|H\right|+\frac{1}{r}\nabla^m r\right)\right)\\ &-2\left|\nabla\tau_1\right|^2+2\left|\tau_1\right|^2\left(\langle H,\lambda\rangle-\langle\nu,\lambda\rangle^2-\frac{1}{|H|}\frac{2}{r^2}\left|\nabla r\right|^2\nu^\beta\lambda_\beta\right)\\ &+2\left|H\right|^2\frac{1}{r^2}\left|\nabla r\right|^2-2\left|H\right|\nabla_i\left(-\frac{1}{r}\nabla^i r\left\langle\nu,\lambda\right\rangle+\frac{2}{|H|}\frac{1}{r}\nabla^i r\left|\lambda\right|^2\right)\\ &+2\left\langle H,\lambda\right\rangle\left(\left|H\right|^2-\left\langle H,\lambda\right\rangle-\frac{2}{r^2}\left|\nabla r\right|^2\right).\end{aligned}$$

As mentioned above, it follows that the loss of $|\tau_1|^2$ in space and time is not too big and Lemma 4.15 is valid for τ_1 instead of τ. Therefore, the integral estimates (cf. Lemma 4.17 and the following corollaries) remain true when replacing τ by τ_1. These estimates imply that along an essential blow-up sequence (p_n, t_n), the solution c becomes planar in the singularity in the sense that

$$\lim_{n\to\infty}\frac{|\tau_1|}{|H|}(p_n,t_n)=0.$$

As for curves in $\mathbb{R}^3$, the solutions are now rescaled along an essential blow-up sequence (cf. (4.23))

$$c_n : S^1\times\left[-\mu_n^2 t_n^2,\mu_n^2\left(T-t_n\right)\right)\to\mathbb{R}^3,\quad c_n\left(\cdot,\tau\right):=\mu_n\left(c\left(\cdot,t\right)-p_n\right),\quad \tau:=\mu_n^2\left(t-t_n\right),$$

where $\mu_n := \left|\mathcal{H}\left(p_n,t_n\right)\right| = \left|H\left(p_n,t_n\right)+\lambda\left(p_n,t_n\right)\right|$. The dilation-invariant estimates lead to the convergence of the rescaled solutions c_n along a subsequence to a limit c_∞ moving by mean curvature flow (cf. Theorem 4.24). The first torsion form $\tau_{1\infty}$ of this limit solution has to vanish identically (cp. Corollary 4.26). Now, Lemma C.1 follows from Lemma 3.6.

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Lebenslauf

Geboren wurde ich, Maren Stroot, am 12.10.1981 in Meppen. Im Jahr 1988 wurde ich in die Grundschule in Lathen eingeschult. Ich wechselte 1992 an das Gymnasium Marianum in Meppen, an dem ich 2001 die Prüfung zur allgemeinen Hochschulreife (Abitur) ablegte.

Im Herbst 2001 began ich ein Mathematikstudium mit dem Nebenfach Logik an der Westfälischen Wilhelms-Universität in Münster. Der Schwerpunkt während meines Hauptstudiums lag im Bereich der analytischen Zahlentheorie. Im Sommer 2006 beendete ich unter der Betreuung von Prof. Dr. Ch. Deninger mein Studium mit der Diplomarbeit *Familien von getwisteten L-Funktionen elliptischer Kurven und Zufallsmatrizen* und erlangte damit den Abschluss Diplom-Mathematiker.

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Hannover, Juni 2010 Maren Stroot